Model Appendix

Industry Structure and the Composition of Men's and Women's Productive Time

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A Firm Optimization

I begin with the derivation of the solution of a representative firm j's ($j = A_m, M_m, S_m$) maximization problem.

$$\max_{l_j^f, l_j^m} p_j Z_j L_j - w_m l_j^m - w_f l_j^f \tag{1a}$$

$$\text{s.t. } L_j = \left[\xi_j \left(l_j^f\right)^{\frac{\eta-1}{\eta}} + (1-\xi_j)\left(\right)l_j^m\left.\right)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{1b}$$

First order conditions with respect to l_j^f and l_j^m yield

$$l_j^f : w_f = p_j Z_j \xi_j \left(l_j^f\right)^{-\frac{1}{\eta}} \tilde{L}_j^{\frac{1}{\eta}} \tag{1c}$$

$$l_j^m : w_m = p_j Z_j (1-\xi_j)\left(l_j^m\right)^{-\frac{1}{\eta}} \tilde{L}_j^{\frac{1}{\eta}} \tag{1d}$$

Combining equations 1c and 1d, we have for firm j that

$$\frac{l_j^m}{l_j^f} = \alpha_j^{-\eta} x^{\eta}, \tag{1e}$$

where $\alpha_j := \frac{\xi}{1-\xi}$ and $x := w_f / w_m$.
Define the female wage bill in sector j as

$$I_j := \frac{w_f l_j^f}{w_f l_j^f + w_m l_j^m} \tag{1f}$$

Using Equation (1e) to replace l_j^m / l_j^f in the female wage bill share, this simplifies to

$$I_j = \frac{1}{1 + \alpha_j^{-\eta} x^{\eta-1}}. \tag{1g}$$

The female wage bill share in both formal sectors is independent of labor units. As in the original model by Ngai and Petrongolo (2017), the sole determinants of the female wage bill share are the wage ratio w_f / w_m and the female specific productivity weight ξ_j.

Next, total effective labor units used in the production of j are computed as a function of female effective labor units, only:

$$L_j = \left[\xi_j\big(l_j^f\big)^{\frac{\eta-1}{\eta}} + (1-\xi_j)\big(\alpha_j^{-\eta}x^\eta\, l_j^f\big)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$
$$= l_j^f\left[\xi_j(1+\alpha_j^{-\eta}x^{\eta-1})\right]^{\frac{\eta}{\eta-1}}.$$

Combining with Equation (1g), this yields

$$L_j = \left(\frac{\xi_j}{I_j(x)}\right)^{\frac{\eta}{\eta-1}} l_j^f. \tag{1h}$$

Equation (1h) can then again be used in order to simplify Equations 1c-1d,

$$w_f = p_j Z_j \xi_j \big(l_j^f\big)^{-\frac{1}{\eta}}\left(l_j^f \xi_j^{\frac{\eta}{\eta-1}} I_j(x)^{\frac{\eta}{1-\eta}}\right)^{\frac{1}{\eta}}$$
$$= p_j Z_j \xi_j^{\frac{\eta}{\eta-1}} I_j(x)^{\frac{1}{1-\eta}}. \tag{1i}$$

Assuming that wages equalize across sectors (not across genders), I can combine Equation (1i) for two sectors:

$$p_i Z_i \xi_i^{\frac{\eta}{\eta-1}} I_i^{\frac{1}{1-\eta}} = w_f = p_j Z_j \xi_j^{\frac{\eta}{\eta-1}} I_j^{\frac{1}{1-\eta}}$$

s.t. the relative prices of two firm commodities equal

$$\boxed{\frac{p_i}{p_j} = \frac{Z_j}{Z_i}\left(\frac{\xi_j}{\xi_i}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_i}{I_j}\right)^{\frac{1}{\eta-1}}}. \tag{1j}$$

B Household Optimization

I now follow with the optimization problem for the representative couple.

$$\max_{\substack{\{c_{j_m}, l^f_{i_h}, l^m_{i_h}\}_{j=A,M,S} \\ l^f_l l^m_l}} \quad ln(c) + \varphi ln(L_l) \tag{2a}$$

s.t.

$$c = \left[\sum_{j=A,M,S} \omega_j c_j^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} \tag{2b}$$

$$c_j = \left[\psi_j (c_{j_m})^{\frac{\sigma-1}{\sigma}} + (1-\psi_j)(c_{i_h})^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{2c}$$

$$c_{i_h} = Z_{i_h} \left[\xi_j (l^f_{i_h})^{\frac{\eta-1}{\eta}} + (1-\xi_j)(l^m_{i_h})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{2d}$$

$$L_l = \left[\xi_l (l^f_l)^{\frac{\eta_l-1}{\eta_l}} + (1-\xi_l)(l^m_l)^{\frac{\eta_l-1}{\eta_l}} \right]^{\frac{\eta_l}{\eta_l-1}} \tag{2e}$$

$$\sum_{j=A,M,S} p_{j_m} c_{j_m} = w_m M^m + w_f M^f \tag{2f}$$

where M^g with $g = f, m$ denotes formal hours of women and men ($M^g = L^g - \sum_{i=A_n,M_n,S_n,l} l^g_i$). I allow for $\eta_l \neq \eta$ in this appendix. The derivatives take the form:

$$c_{j_m} : \frac{\partial U}{\partial c_{j_m}} = \lambda p_{j_m}, \qquad j = A, M, S \tag{3a}$$

$$l^g_{i_h} : \frac{\partial U}{\partial c_{i_h}} \frac{\partial c_{i_h}}{\partial l^g_{i_h}} = \lambda w_g, \qquad j = A, M, S \tag{3b}$$

$$l^g_l : \frac{\partial U}{\partial l^g_l} = \lambda w_g, \qquad g = f, m. \tag{3c}$$

B.1 Formal vs. Traditional Price Ratios

On the traditional production side for commodities $i_h = A_h, M_h, S_h$, compute the marginal rate of technical substitution by combining Equation for men and women (3b):

$$\frac{\frac{\delta c_{i_h}}{\delta l^f_{i_h}}}{\frac{\delta c_{i_h}}{\delta l^m_{i_h}}} = \frac{w_f}{w_m}, \tag{4a}$$

Since

$$\frac{\delta c_{i_h}}{\delta l^f_{i_h}} = Z_{i_h}\xi_{i_h}\left(l^f_{i_h}\right)^{-\frac{1}{\eta}} L^{\frac{1}{\eta}}_{i_h} \tag{4b}$$

$$\frac{\delta c_{i_h}}{\delta l^m_{i_h}} = Z_{i_h}(1-\xi_{i_h})\left(l^m_{i_h}\right)^{-\frac{1}{\eta}} L^{\frac{1}{\eta}}_{i_h}, \tag{4c}$$

we get male labor hours in traditional production as a function of female traditional hours:

$$\left(\frac{l^f_{i_h}}{l^m_{i_h}}\right)^{-\frac{1}{\eta}} \alpha_{i_h} = x$$

$$l^m_{i_h} = \alpha^{-\eta}_{i_h} x^\eta l^f_{i_h} \tag{4d}$$

In this model, it will be useful to define and derive the ratio of lifetime female over total household earnings, since couples will not start their working life at the same time. I define the female lifetime earnings share as

$$I_{i_h} := \frac{w_f l^f_{i_h}}{w_f l^f_{i_h} + w_m l^m_{i_h}}. \tag{5a}$$

Using Equation (4d) to replace male by female hours in the above denominator yields

$$I_{i_h} = \frac{w_f l^f_{i_h}}{w_f l^f_{i_h}\left(1 + \underbrace{\frac{w_m}{w_f}}_{=x^{-1}} \alpha^{-\eta}_{i_h} x^\eta\right)}$$

Thus,

$$\boxed{I_{i_h} = \frac{1}{1+\alpha^{-\eta}_{i_h} x^{\eta-1}}}. \tag{5b}$$

Also, use female hours from Equation (4d) to replace $l^m_{i_h}$ in the CES for the total supply of labor in traditional production (L_{i_h}) in Equation (2d), which yields

$$L_{i_h} = \left[\xi_{i_h}\left(l^f_{i_h}\right)^{\frac{\eta-1}{\eta}} + (1-\xi_{i_h})\left(\alpha_h^{-\eta}x^\eta l^f_{i_h}\right)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{6}$$

$$\boxed{\frac{L_{i_h}}{l^f_{i_h}} = \left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}}, j = A, M, S} \tag{7}$$

Now define the implicit price of the household commodity: Analogue to the derivatives wrt c_{j_m} in Equation (3a), I require that

$$\frac{\delta U}{\delta c_{i_h}} = p_{i_h}\lambda \tag{8}$$

Rearrange the derivative wrt $l^f_{i_h}$ in Equation (3b) for the shadow price λ:

$$\begin{aligned}
\lambda &= \frac{\delta U}{\delta c_{i_h}}\frac{\delta c_{i_h}}{\delta l^f_{i_h}}\frac{1}{w_f}, \text{ s.t.} \\
\frac{\delta U}{\delta c_{i_h}}\frac{1}{p_{i_h}} &= \frac{\delta U}{\delta c_{i_h}}\frac{\delta c_{i_h}}{\delta l^f_{i_h}}\frac{1}{w_f} \\
p_{i_h} &= w_f\left(\frac{\delta c_{i_h}}{\delta l^f_{i_h}}\right)^{-1}
\end{aligned} \tag{9a}$$

Replace total labor in traditional production in the derivative $\frac{\delta c_{i_h}}{\delta l^f_{i_h}}$ (see Equation (4b)) using the expression we derived for total relative to female labor in traditional production in Equation (7):

$$\begin{aligned}
\frac{\delta c_{i_h}}{\delta l^f_{i_h}} &= Z_{i_h}\xi_{i_h}\left(l^f_{i_h}\right)^{-\frac{1}{\eta}}\left[\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} l^f_{i_h}\right]^{\frac{1}{\eta}} \\
\frac{\delta c_{i_h}}{\delta l^f_{i_h}} &= Z_{i_h}\xi_{i_h}\left(l^f_{i_h}\right)^{-\frac{1}{\eta}}\left(L_{jn}\right)^{\frac{1}{\eta}} = Z_{i_h}\xi_{i_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}}
\end{aligned} \tag{9b}$$

Replace Equation (9b) in Equation (9a) to derive the implicit wage condition on the traditional production sector:

$$p_{i_h} = w_f \left(Z_{i_h} \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \right)^{-1}$$
$$w_f = p_{i_h} Z_{i_h} \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}} . \tag{9c}$$

This model assumes that wages equalize across sectors, but not across genders. Thus make use of the equation derived on the firm side relating the price of firm commodities $p_{A_m}, p_{M_m}, p_{S_m}$ to wages (see Eq. 1i), to infer relative prices of firm commodities (j_m) to the implicit price of household commodities (i_h):

$$p_{i_h} Z_{i_h} \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}} = p_{j_m} Z_{j_m} \xi_{j_m}^{\frac{\eta}{\eta-1}} I_{j_m}^{\frac{1}{1-\eta}}$$

$$\boxed{\frac{p_{i_h}}{p_{j_m}} = \frac{Z_{j_m}}{Z_{i_h}} \left(\frac{\xi_{j_m}}{\xi_{i_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}}} \tag{10a}$$

or vice versa:

$$\boxed{\frac{p_{j_m}}{p_{i_h}} = \frac{Z_{i_h}}{Z_{j_m}} \left(\frac{\xi_{i_h}}{\xi_{j_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_h}}\right)^{\frac{1}{\eta-1}}}, \tag{10b}$$

for $i, j = A, M, S$.

Also, the relative price of two different kinds of household commodities (that I now denote by $i, k = A, M, S$) equals

$$p_{i_h} Z_{i_h} \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}} = p_{k_h} Z_{k_h} \xi_{k_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{k_h}}\right)^{\frac{1}{\eta-1}}$$

$$\boxed{\frac{p_{i_h}}{p_{k_h}} = \frac{Z_{k_h}}{Z_{i_h}} \left(\frac{\xi_{i_h}}{\xi_{k_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_h}}{I_{k_h}}\right)^{\frac{1}{\eta-1}}} \tag{11a}$$

B.2 Traditional commodities and leisure prices

For the price of leisure relative to the traditional consumption good, combining the first order conditions wrt l_l^f & l_l^m, we have that

$$
\begin{aligned}
\frac{\frac{\delta L_l}{\delta l_l^f}}{\frac{\delta L_l}{\delta l_l^m}} &= \frac{w_f}{w_m} = x \\
\frac{\xi_l \big(l_l^f\big)^{-\frac{1}{\eta_l}} L_l^{\frac{1}{\eta_l}}}{(1-\xi_l)\big(l_l^m\big)^{-\frac{1}{\eta_l}} L_l^{\frac{1}{\eta_l}}} &= \frac{\xi_l}{1-\xi_l}\left(\frac{l_l^m}{l_l^f}\right)^{\frac{1}{\eta_l}} = x \\
\frac{l_l^m}{l_l^f} &= \alpha_l^{-\eta_l} x^{\eta_l} \qquad (12)
\end{aligned}
$$

Apply the same trick as before, i.e. define $I_l(x, s)$ as the female wage bill share in leisure, during working life:

$$
I_l := \frac{w_f l_l^f}{w_f l_l^f + w_m l_l^m}. \qquad (13a)
$$

Then I_l reduces to

$$
I_l = \frac{1}{1 + \alpha_l^{-\eta_l} x^{\eta_l - 1}} \qquad (13b)
$$

Replace male leisure time by female leisure time in aggregate leisure L_l (see Equation (2e)) yields

$$
\begin{aligned}
L_l &= \left[\xi_l (l_l^f)^{\frac{\eta_l - 1}{\eta_l}} + (1-\xi_l)(\alpha_l^{-\eta_l} x^{\eta_l} l_l^f)^{\frac{\eta_l - 1}{\eta_l}}\right]^{\frac{\eta_l}{\eta_l - 1}} \\
L_l &= l_l^f \left(\frac{\xi_l}{I_l(x, s)}\right)^{\frac{\eta_l}{\eta_l - 1}} \qquad (14)
\end{aligned}
$$

Analogue to the traditional consumption good, require that $\frac{\delta U}{\delta l_l^f} = \lambda p_l$. Rearrange the derivative wrt l_l^f in Equation (3c) for λ and substitute:

$$
\begin{aligned}
\lambda = \frac{\delta U}{\delta l_l^f}\frac{1}{p_l} &= \frac{1}{w_f}\frac{\delta U}{\delta L_l}\frac{\delta L_l}{\delta l_l^f} \\
p_l &= w_f \left(\frac{\delta L_l}{\delta l_l^f}\right)^{-1} \qquad (15)
\end{aligned}
$$

Compute the derivative of aggregate leisure wrt female leisure time, and replace using Equation (14):

$$\frac{\delta L_l}{\delta l_l^f} = \xi_l^{\frac{\eta_l}{\eta_l - 1}} \left(\frac{1}{I_l(x,s)}\right)^{\frac{1}{\eta_l - 1}} \tag{16a}$$

Then the leisure price equals

$$p_l = w_f \left(\xi_l^{\frac{\eta_l}{\eta_l - 1}} \left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l - 1}}\right)^{-1} \tag{16b}$$

$$w_f = p_l \xi_l^{\frac{\eta_l}{\eta_l - 1}} \left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l - 1}} \tag{16c}$$

Since wages clear across sectors, we can equate Equation (16c) and (9c) to get the price of leisure relative to the price of a household commodity:

$$p_l \xi_l^{\frac{\eta_l}{\eta_l - 1}} \left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l - 1}} = p_{i_h} Z_{i_h} \xi_{i_h}^{\frac{\eta}{\eta - 1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta - 1}}$$

$$\boxed{\frac{p_l}{p_{i_h}} = Z_{i_h} \xi_{i_h}^{\frac{\eta}{\eta - 1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta - 1}} \xi_l^{\frac{\eta_l}{1 - \eta_l}} \left(\frac{1}{I_l}\right)^{\frac{1}{1 - \eta_l}}} \tag{17a}$$

$$\boxed{\frac{p_{i_h}}{p_l} = Z_{i_h} \xi_{i_h}^{\frac{\eta}{1 - \eta}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{1 - \eta}} \xi_l^{\frac{\eta_l}{\eta_l - 1}} \left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l - 1}}} \tag{17b}$$

And for the price ratio of leisure vs firm commodities, we have

$$p_l \xi_l^{\frac{\eta_l}{\eta_l - 1}} \left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l - 1}} = p_{j_m} Z_{j_m} \xi_{j_m}^{\frac{\eta}{\eta - 1}} I_{j_m}^{\frac{1}{1 - \eta}}$$

$$\boxed{\frac{p_l}{p_{j_m}} = Z_{j_m} \xi_{j_m}^{\frac{\eta}{\eta - 1}} \left(\frac{1}{I_{j_m}}\right)^{\frac{1}{\eta - 1}} \left(\frac{1}{\xi_l}\right)^{\frac{\eta_l}{\eta_l - 1}} I_l^{\frac{1}{\eta_l - 1}}} \tag{17c}$$

B.3 Relative Expenditures

Formal versus traditional expenditures

Derive $MRS_{c_{jm},c_{i_h}}$: We combine derivatives wrt c_{jm} and $l^f_{i_h}$ (see Equations (3a) and (3b)), to have marginal rates of substitution between formal vs traditional services:

$$MRS_{c_{jm},c_{i_h}} = \frac{\frac{\delta U}{\delta c_{jm}}}{\frac{\delta U}{\delta c_{i_h}}} = \frac{\lambda p_{jm}}{\lambda w_f}\frac{\delta c_{i_h}}{\delta l^f_{i_h}} = \frac{p_{jm}}{w_f}\frac{\delta c_{i_h}}{\delta l^f_{i_h}}$$

Note that the derivative expansions needed from the chain rule on the LHS cancel out, such that

$$\frac{\frac{\delta U}{\delta c}\frac{\delta c}{\delta c_j}\frac{\delta c_j}{\delta c_{jm}}}{\frac{\delta U}{\delta c}\frac{\delta c}{\delta c_j}\frac{\delta c_j}{\delta c_{i_h}}} = \frac{\frac{\delta c_j}{\delta c_{jm}}}{\frac{\delta c_j}{\delta c_{i_h}}} = \frac{\psi_j c_{jm}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{(1-\psi_j) c_{i_h}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}$$

$$\frac{\psi_j}{1-\psi_j}\left(\frac{c_{jm}}{c_{i_h}}\right)^{-\frac{1}{\sigma}} = \frac{p_{jm}}{w_f}\frac{\delta c_{i_h}}{\delta l^f_{i_h}}$$

We know w_f from Equation (9a) that related the implicit price of traditional services to wages. Substitute and get

$$\frac{\psi_j}{1-\psi_j}\left(\frac{c_{jm}}{c_{i_h}}\right)^{-\frac{1}{\sigma}} = \frac{p_{jm}}{p_{i_h}}\frac{\frac{\delta c_{i_h}}{\delta l^f_{i_h}}}{\frac{\delta c_{i_h}}{\delta l^f_{i_h}}} = \frac{p_{jm}}{p_{i_h}}$$

$$\left(\frac{c_{jm}}{c_{i_h}}\right)^{\frac{1}{\sigma}} = \frac{p_{i_h}}{p_{jm}}\frac{\psi_j}{1-\psi_j}$$

$$\boxed{\frac{c_{jm}}{c_{i_h}} = \left(\frac{p_{i_h}}{p_{jm}}\right)^{\sigma}\left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma}} \tag{18}$$

Relative expenditures on formal relative to household commodities then are:

$$\boxed{E_{jmh} = \frac{p_{jm}c_{jm}}{p_{i_h}c_{i_h}} = \left(\frac{p_{i_h}}{p_{jm}}\right)^{\sigma-1}\left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma_j}} \tag{19a}$$

Making use of our results for relative prices p_{i_h}/p_{j_m} in Equation (10a)

$$\frac{p_{i_h}}{p_{j_m}} = \frac{Z_{j_m}}{Z_{i_h}} \left(\frac{\xi_{j_m}}{\xi_{i_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \tag{19b}$$

and defining $\hat{Z}_{j_{mh}} := \frac{Z_{j_m}}{Z_{i_h}} \left(\frac{\psi_j}{1-\psi_j}\right)^{\frac{\sigma}{\sigma-1}}$

$$E_{j_{mh}} = \left[\frac{Z_{j_m}}{Z_{i_h}} \left(\frac{\xi_{j_m}}{\xi_{i_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}}\right]^{\sigma-1} \left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma}$$

Eventually, the relative expenditure share of formal to household commodity j = A,M,S equals

$$\boxed{E_{j_{mh}} = \hat{Z}_{j_{mh}}^{\sigma-1} \left[\left(\frac{\xi_{j_m}}{\xi_{i_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}}\right]^{\sigma-1}}. \tag{19c}$$

Expenditures across commodity kinds

For firm commodities c_{j_m}, c_{i_m}, derive $MRS_{c_{j_m},c_{i_m}}$. Combine the derivatives wrt c_{j_m} and c_{i_m} (see Equation 3a):

$$\begin{aligned}
MRS_{c_{j_m},c_{i_m}} = \frac{p_{j_m}}{p_{i_m}} &= \frac{\frac{\delta U}{\delta c}\frac{\delta c}{\delta c_j}\frac{\delta c_j}{\delta c_{j_m}}}{\frac{\delta U}{\delta c}\frac{\delta c}{\delta c_i}\frac{\delta c_i}{\delta c_{i_m}}} = \frac{\omega_j}{\omega_i}\frac{c_j^{-\frac{1}{\epsilon}}\psi_j c_{j_m}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{c_i^{-\frac{1}{\epsilon}}\psi_i c_{i_m}^{-\frac{1}{\sigma}} c_i^{\frac{1}{\sigma}}} \\
\frac{p_{j_m}}{p_{i_m}} &= \frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\left(\frac{c_i}{c_j}\right)^{\frac{1}{\epsilon}}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{1}{\sigma}}\left(\frac{c_j}{c_{j_m}}\right)^{\frac{1}{\sigma}} \\
\left(\frac{p_{j_m}}{p_{i_m}}\right)^{\epsilon} &= \frac{c_{i_m}}{c_{j_m}}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\epsilon-\sigma}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\sigma-\epsilon}{\sigma}}\left(\frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\right)^{\epsilon}
\end{aligned}$$

$$\boxed{\frac{c_{j_m}}{c_{i_m}} = \left(\frac{p_{i_m}}{p_{j_m}}\right)^{\epsilon}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\epsilon-\sigma}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\sigma-\epsilon}{\sigma}}\left(\frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\right)^{\epsilon}} \tag{20a}$$

and

$$\frac{c_{i_m}}{c_{j_m}} = \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\epsilon} \left(\frac{c_{i_m}}{c_i}\right)^{\frac{\sigma-\epsilon}{\sigma}} \left(\frac{c_{j_m}}{c_j}\right)^{\frac{\epsilon-\sigma}{\sigma}} \left(\frac{\omega_i}{\omega_j}\frac{\psi_i}{\psi_j}\right)^{\epsilon} \tag{20b}$$

Problematic: The c_i, c_j in Equation (20b). Consider the inverse relative demand c_{i_m}/c_{i_n}

$$\frac{c_{i_m}}{c_{i_n}} = \left(\frac{p_{i_n}}{p_{i_m}}\right)^{\sigma} \left(\frac{\psi_i}{1-\psi_i}\right)^{\sigma}$$

from 18 to derive c_i as a function of c_{i_m}:

$$c_i = \left[\psi_i c_{i_n}^{\frac{\sigma-1}{\sigma}} + (1-\psi_i) c_{i_m}^{\frac{\sigma-1}{\sigma}} \left(\frac{p_{i_m}}{p_{i_n}}\frac{1-\psi_i}{\psi_i}\right)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}}$$

$$\frac{c_i}{c_{i_m}} = \psi_i^{\frac{\sigma}{\sigma-1}} \Bigg[1 + \underbrace{\left(\frac{1-\psi_i}{\psi_i}\right)^{\sigma} \left(\frac{p_{i_m}}{p_{i_n}}\right)^{\sigma-1}}_{=E_{i_{mh}}^{-1}}\Bigg]^{\frac{\sigma}{\sigma-1}} \tag{21a}$$

Note that the inner part of Equation (21a) equals the inverse of $E_{i_{mh}}$ in Equation (19a), so that one can write

$$\frac{c_i}{c_{i_m}} = \psi_i^{\frac{\sigma}{\sigma-1}} \left(1 + \frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma}{\sigma-1}} \tag{21b}$$

This equation holds for $i = A, M, S$, i.e. we also get

$$\frac{c_j}{c_{j_m}} = \psi_j^{\frac{\sigma}{\sigma-1}} \left(1 + \frac{1}{E_{j_{mh}}}\right)^{\frac{\sigma}{\sigma-1}} \tag{22}$$

Plug into Equation (20a), which yields

$$\frac{c_{j_m}}{c_{i_n}} = \left(\frac{p_{i_n}}{p_{j_m}}\right)^{\epsilon} \left(1 + \frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\epsilon}{\sigma-1}} \left(1 + \frac{1}{E_{j_{mh}}}\right)^{\frac{\epsilon-\sigma}{\sigma-1}} \left(\frac{\omega_i}{\omega_j}\right)^{\epsilon} \psi_i^{\sigma\frac{\epsilon-1}{1-\sigma}} \psi_j^{\sigma\frac{1-\epsilon}{1-\sigma}} \tag{23}$$

and since it will be useful to have everything in terms of relative expenditures, again write

$$\frac{p_{j_m} c_{j_m}}{p_{i_m} c_{i_m}} = E_{j_m i_m} = \left(\frac{p_{i_m}}{p_{j_m}}\right)^{\epsilon-1} \left(1 + \frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\epsilon}{\sigma-1}} \left(1 + \frac{1}{E_{j_{mh}}}\right)^{\frac{\epsilon-\sigma}{\sigma-1}} \left(\frac{\omega_i}{\omega_j}\right)^{\epsilon} \psi_i^{\sigma \frac{\epsilon-1}{1-\sigma}} \psi_j^{\sigma \frac{1-\epsilon}{1-\sigma}} \tag{24a}$$

Again, make use of the relative price equation from the market side (see Equation (1j)) to substitute for p_s/p_g and define

$$\hat{Z}_{j_m i_m} := \frac{Z_{j_m}}{Z_{i_m}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{\epsilon}{\epsilon-1}} \psi_i^{\frac{\sigma}{1-\sigma}} \psi_j^{\frac{\sigma}{\sigma-1}} \tag{24b}$$

to get

$$E_{j_m i_m} = \hat{Z}_{j_m i_m}^{\epsilon-1} \left[\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_m}}\right)^{\frac{1}{1-\eta}} \right]^{\epsilon-1} \left(1 + \frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\epsilon}{\sigma-1}} \left(1 + \frac{1}{E_{j_{mh}}}\right)^{\frac{\epsilon-\sigma}{\sigma-1}} \tag{24c}$$

Leisure:
$MRS_{L_l, c_{j_m}}$. Equate marginal utilities, set equal to price ratio. Note that

$$\frac{\delta U}{\delta L_l} = \lambda w_f \left(\frac{\delta L_l}{\delta L_{fl}}\right)^{-1} = \lambda p_l$$

by our definition of the implicit leisure price (see Equation (15)). Thus

$$\begin{aligned} \frac{\frac{\delta U}{\delta c_{j_m}}}{\frac{\delta U}{\delta L_l}} &= \frac{p_{j_m}}{p_l} \\ \frac{\frac{1}{c-\bar{c}} \omega_j c_j^{-\frac{1}{\epsilon}} c^{\frac{1}{\epsilon}} \psi_j c_{j_m}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{\frac{\varphi}{L_l}} &= \frac{p_{j_m}}{p_l} \\ \frac{1}{\varphi} \frac{c}{c-\bar{c}} \omega_j \psi_j \left(\frac{c}{c_j}\right)^{\frac{1-\epsilon}{\epsilon}} \left(\frac{c_j}{c_{j_m}}\right)^{\frac{1-\sigma}{\sigma}} &= \frac{p_{j_m} c_{j_m}}{p_l L_l} \end{aligned} \tag{25}$$

We know the relations c_j/c_{j_m} and c_i/c_{i_m} from Equation (22) and Equation

(21b)

$$\frac{c_j}{c_{j_m}} = \psi_j^{\frac{\sigma}{\sigma-1}} \left(1 + \frac{1}{E_{j_m j_h}}\right)^{\frac{\sigma}{\sigma-1}}$$

Also,

$$\begin{aligned}
c &= \left[\sum_{i=A,M,S} \omega_i (c_i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \\
c &= \left[\omega_j (c_j)^{\frac{\epsilon-1}{\epsilon}} \sum_{i=A,M,S} \frac{\omega_i}{\omega_j} \left(\frac{c_i}{c_j}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}, j = A, M, S \\
c &= \left[\omega_j (c_j)^{\frac{\epsilon-1}{\epsilon}} \sum_{i=A,M,S} \frac{\omega_i}{\omega_j} \left(\frac{c_i}{c_{i_m}} \frac{c_{i_m}}{c_{j_m}} \frac{c_{j_m}}{c_j}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \\
\frac{c}{c_j} &= \left[\omega_j \sum_{i=A,M,S} \frac{\omega_i}{\omega_j} \left(\frac{c_i}{c_{i_m}} \frac{c_{i_m}}{c_{j_m}} \frac{c_{j_m}}{c_j}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \qquad (26a)
\end{aligned}$$

Let's simplify the expression containing multiple relative demands, using Equation (20b):

$$\begin{aligned}
\frac{c_{i_m}}{c_{j_m}} \frac{c_i}{c_{i_m}} \frac{c_{j_m}}{c_j} &= \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\epsilon} \left(\frac{c_{i_m}}{c_i}\right)^{\frac{\sigma-\epsilon}{\sigma}} \left(\frac{c_{j_m}}{c_j}\right)^{\frac{\epsilon-\sigma}{\sigma}} \left(\frac{\omega_i}{\omega_j} \frac{\psi_i}{\psi_j}\right)^{\epsilon} \frac{c_i}{c_{i_m}} \frac{c_{j_m}}{c_j} \\
&= \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\epsilon} \left(\frac{c_i}{c_{i_m}}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{c_{j_m}}{c_j}\right)^{\frac{\epsilon}{\sigma}} \left(\frac{\omega_i}{\omega_j} \frac{\psi_i}{\psi_j}\right)^{\epsilon} \\
\frac{c_i}{c_j} &= \left(\hat{Z}_{j_m i_m} \left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_m}}{I_{j_m}}\right)^{\frac{1}{\eta-1}}\right)^{-\epsilon} \left(1 + \frac{1}{E_{i_m h}}\right)^{\frac{\epsilon}{\sigma-1}} \left(1 + \frac{1}{E_{j_m h}}\right)^{\frac{\epsilon}{1-\sigma}} \left(\frac{\omega_j}{\omega_i}\right)^{\frac{\epsilon}{\epsilon-1}} \qquad (27a)
\end{aligned}$$

Remember that we had defined in Equation (24c)

$$\begin{aligned}
E_{j_m i_m} &= \left[\hat{Z}_{j_m i_m} \left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_m}}\right)^{\frac{1}{1-\eta}}\right]^{\epsilon-1} \left(1 + \frac{1}{E_{i_m h}}\right)^{\frac{\sigma-\epsilon}{\sigma-1}} \left(1 + \frac{1}{E_{j_m h}}\right)^{\frac{\epsilon-\sigma}{\sigma-1}} \\
E_{j_m i_m}^{\frac{1}{\epsilon-1}} &= \left[\hat{Z}_{j_m i_m} \left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_m}}\right)^{\frac{1}{1-\eta}}\right] \left(1 + \frac{1}{E_{i_m h}}\right)^{\frac{\sigma-\epsilon}{(\sigma-1)(\epsilon-1)}} \left(1 + \frac{1}{E_{j_m h}}\right)^{\frac{\epsilon-\sigma}{(\sigma-1)(\epsilon-1)}} \\
E_{j_m i_m}^{-\frac{\epsilon}{\epsilon-1}} &= \left[\hat{Z}_{j_m i_m} \left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_m}}\right)^{\frac{1}{1-\eta}} \left(1 + \frac{1}{E_{i_m h}}\right)^{\frac{\sigma-\epsilon}{(\sigma-1)(\epsilon-1)}} \left(1 + \frac{1}{E_{j_m h}}\right)^{\frac{\epsilon-\sigma}{(\sigma-1)(\epsilon-1)}}\right]^{-\epsilon} \qquad (27b)
\end{aligned}$$

Thus, Equation (27a) simplifies, because

$$= \Bigg[\underbrace{\left(\hat{Z}_{j_m i_m} \left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_m}}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \right)^{\epsilon} \left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\epsilon}{\sigma-1}\frac{\epsilon}{\epsilon-1}} \left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\epsilon-\sigma}{\sigma-1}\frac{\epsilon}{\epsilon-1}}}_{=E_{j_m i_m}^{\frac{\epsilon}{\epsilon-1}}} \Bigg]^{-1}$$

$$\times \left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\epsilon}{\epsilon-1}} \left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\epsilon}{1-\epsilon}} \left(\frac{\omega_j}{\omega_i}\right)^{\frac{\epsilon}{\epsilon-1}}$$

$$= \left(E_{j_m i_m}\right)^{\frac{\epsilon}{1-\epsilon}} \left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\epsilon}{\epsilon-1}} \left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\epsilon}{1-\epsilon}} \left(\frac{\omega_j}{\omega_i}\right)^{\frac{\epsilon}{\epsilon-1}}$$

$$\boxed{\frac{c_i}{c_j} = \left(E_{j_m i_m} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\epsilon}{1-\epsilon}} \left(\frac{\omega_j}{\omega_i}\right)^{\frac{\epsilon}{\epsilon-1}}} \tag{28}$$

We then have that

$$\frac{c}{c_j} = \left[\omega_j \sum_{i=A,M,S} \frac{\omega_i}{\omega_j} \left(\frac{c_i}{c_j}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$$

$$\frac{c}{c_j} = \omega_j^{\frac{\epsilon}{\epsilon-1}} \left[\sum_{i=A,M,S} \left(E_{j_m i_m} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{-1}\right]^{\frac{\epsilon}{\epsilon-1}}$$

$$c = c_j \omega_j^{\frac{\epsilon}{\epsilon-1}} E_j^{\frac{\epsilon}{\epsilon-1}} \tag{29a}$$

where $E_j := \frac{E_{j_{mh}}}{1+E_{j_{mh}}} \sum_{i=A,M,S} \left(E_{i_m j_m} \frac{1+E_{i_{mh}}}{E_{i_{mh}}}\right)$.

The expression in Equation (25) then becomes

$$\frac{p_{j_m} c_{j_m}}{p_l L_l} = \frac{1}{\varphi} \frac{c}{c-\bar{c}} \omega_j \psi_j \left(\omega_j^{\frac{\epsilon}{\epsilon-1}} E_j^{\frac{\epsilon}{\epsilon-1}}\right)^{\frac{1-\epsilon}{\epsilon}} \left(\frac{c_j}{c_{j_m}}\right)^{\frac{1-\sigma}{\sigma}}$$

$$\frac{p_{j_m} c_{j_m}}{p_l L_l} = \frac{1}{\varphi} \frac{c}{c-\bar{c}} \psi_j E_j^{-1} \left(\frac{c_j}{c_{j_m}}\right)^{\frac{1-\sigma}{\sigma}}$$

$$\frac{p_{j_m} c_{j_m}}{p_l L_l} = \frac{1}{\varphi} \frac{c}{c-\bar{c}} \psi_j E_j^{-1} \left(\psi_j^{\frac{\sigma}{\sigma-1}} \left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\sigma}{\sigma-1}}\right)^{\frac{1-\sigma}{\sigma}} \tag{30}$$

$$E_{j_m l} = \frac{1}{\varphi}\frac{c}{c-\bar{c}}E_j^{-1}\left(1+\frac{1}{E_{j_{mh}}}\right)^{-1} \tag{31}$$

$$E_{l j_m} = \varphi\frac{c-\bar{c}}{c}E_j\left(1+\frac{1}{E_{j_{mh}}}\right) \tag{32}$$

B.4 Closing the model

From the time constraint, it is known that total available time L_f is to be spent in the production of commodities or as leisure time. Time allocations also enter the budget constraint. For both constraints, it is possible to derive women's relative time spent in non-market production of services as a function of relative expenditures and the female wage bill shares. These in turn are determined through the wage ratio x. As in the original paper by Ngai and Petrongolo (2017), I derive and use these conditions to solve for a numerical approximation of the optimal wage ratio x in a root-finding algorithm.

B.4.1 Relative hours and the time constraint

The total time constraint of women imposes that

$$\sum_{j=A,M,S} \sum_{k=m,n} l^f_{jk} = L^f - l^f_l, \tag{33a}$$

and market clearing conditions require that demand of any firm or traditional commodity $j = A_h, A_m, M_h, M_m, S_h, S_m, l$ equals it supply, i.e. $p_j c_j = p_j A_j L_j$. Divide this by expenditures on any other commodity k, and replace the CES aggregates for total time inputs L_j by female hours as given in Equation (1h). This yields

$$E_{jk} = \frac{p_j A_j L_j}{p_k A_k L_k} = \frac{I_k}{I_j} \frac{l^f_j}{l^f_k}. \tag{33b}$$

So, relative hours are given by

$$\boxed{\frac{l^f_j}{l^f_k} = E_{jk} \frac{I_j}{I_k}.} \tag{33c}$$

In the compuational solver, I set k = S_h, so that

$$\frac{l^f_j}{l^f_{S_h}} = E_{jS_h} \frac{I_j}{I_{S_h}}$$

The time constraint Equation (33a) can then be rewritten as

$$\frac{l^f_{S_h}}{L^f} = \frac{1}{\sum_{j=A,M,S} \sum_{k=m,n} E_{j_k S_n} \frac{I_{j_k}}{I_{S_h}}} \tag{34}$$

B.4.2 The budget constraint

Using the definitions for implicit prices on time spent in non-market services, non-market goods and leisure, I rewrite the budget constraint and normalize by expenditures on non-market produced services, $p_{S_n} c_{S_n}$:

$$\sum_{j=A,M,S} \sum_{k=m,n} E_{j_k S_n} + E_{lS_n} = \frac{w_f L^f + w_m L^m}{p_{S_n} c_{S_n}} \tag{35a}$$

It is then possible to derive the share of female time in non-market services. Define the female wage bill share $I(x) := \frac{w_f L_f}{w_f L_f + w_m L_m}$ and replace the implicit price of the non-market service commodity as given in Equation (1i) yields

$$\frac{l^f_{S_n}}{L^f} = \frac{1}{I_L \sum_{j=A,M,S} \sum_{k=m,n} \frac{E_{j_k S_n}}{I_{S_n}}} \tag{35b}$$

The root-finding algorithm solves for the optimal wage ratio x such that both, Equations (34) (35b) hold. Given x, I can then retrieve the time allocations of women and men.

References

Ngai, L. R. and Petrongolo, B. (2017), 'Gender gaps and the rise of the service economy', *American Economic Journal: Macroeconomics* **9**(4), 1–44.