Model Appendix

Structural Change, Gender Gaps and Educational Choice

(For online publication only)

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A Benchmark Model

A.1 Preferences for schooling and implications

A.1.1 Schooling as a nuisance

As is the case in Restuccia and Vandenbroucke (2014, p.831), in general I need a preference for schooling. Otherwise, the schooling choices that women and men face collapse to one where both agents just maximize lifetime income, and I cannot capture non-pecuniary costs and benefits of schooling. To see this, consider the following lifetime-income maximization problem:

$$\bar{s} := \max_{s_g} \int_{t=s_g}^{T} e^{-\rho t} dt H(s_g) M^g w_g. \tag{1a}$$

Again, let $d(s_g) := \int_{t=s_g}^{T} e^{-\rho t} dt$. The schooling choice that maximizes lifetime income, conditional on hours and wages, is denoted by s^*. This is the level of schooling that solves the first order condition to Equation (1a):

$$d(s_g)H'(s_g) = d'(s_g)H(s_g). \tag{1b}$$

This is the conditional marginal effect of schooling on lifetime income (holding hours in formal work and wages constant). It captures the trade-off that results from a marginal increase in schooling years: The raise in human capital $H(s_g)$ (and thus productive wages), against the reduction of the remaining lifetime during which agents can engage in formal work. For $s_g = s^*$ such that Equation (1b) holds, the two perfectly trade off and the term $d(s_g)H(s_g)$ is maximized.

Assume that the discount rate $\rho = 0.04$. This is reasonable and as in Restuccia and Vandenbroucke (2014). Take WDI data on life expectancy T in low income countries (46 years), middle (60 years) and high income countries (69 years), and assume that agents work until the end of their life. Finally, parametrize $H(s_g)$ as in Bils and Klenow (2000), with $\nu = 0.58, \zeta = 0.32$. Then, the schooling years that solve Condition (1b) numerically are 18.5 years in low, 23 in middle and 25.8 years in high income countries. This is substantially more than the schooling years I observe in the data: 6.4 (5.2) years for men (women) in low, 9.1 (8.7) years in middle and 11.4 (11.2) years in high income countries.

A.1.2 Implications

Combining Equations (5b) and (5d) from the main text yields

$$\beta_g e^{-\rho s_g} = -M^g \left[\frac{H'(s_g)}{H(s_g)} + \frac{d'(s_g)}{d(s_g)} \right] \frac{\partial U}{\partial c_{jH}} \frac{\partial c_{jH}}{\partial l^g_{jH}}. \tag{2}$$

On the left-hand side is the marginal utility of schooling. The right hand side is the marginal cost of schooling. Here, hours in formal work M^g and the last two terms are positive by construction. The middle term

$$\hat{H}(s_g) := \frac{H'(s_g)}{H(s_g)} + \frac{d'(s_g)}{d(s_g)}. \tag{3}$$

is a rewritten version of Equation (1b). Whether this conditional marginal effect $\hat{H}(s_g)$ is positive or negative depends on the sign of β_g. For $\beta_g < 0$, schooling is a nuisance and it must be that $\hat{H}(s_g) > 0$ for Equation (2) to hold. That is, the term $d(s_g)H(s_g)$ is increasing in s_g. On the contrary, for $\beta_g > 0$, schooling is a nuisance and it must be that $\hat{H}(s_g) < 0$ for Equation (2) to hold. Then, the term $d(s_g)H(s_g)$ is decreasing in s_g. As I show above for reasonable parametrizations of ρ, T and $H(s_g)$, schooling years in the data are lower than the levels of schooling that maximize life-time income, holding hours fixed (for the details, refer to Appendix A.1). this speaks for a case where $\beta_g < 0$.

A.2 Firm Optimization

I begin with the derivation of the solution of a representative firm j's ($j = A_m, M_m, S_m$) maximization problem.

$$\max_{L^f_j, L^m_j} \; p_j Z_j L_j - w_m(T - s_m)H(s_m)L^m_j - w_f(T - s_f)H(s_f)L^f_j \tag{4a}$$

s.t.

$$L_j = \left[\xi_j \big(H(s_f)(T - s_f)L^f_j\big)^{\frac{\eta-1}{\eta}} + (1 - \xi_j)\big(H(s_m)(T - s_m)L^m_j\big)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{4b}$$

First order conditions with respect to L^f_j and L^m_j yield

$$L^f_j : w_f = p_j Z_j * \xi_j \big((T - s_f)H(s_f)\big)^{-\frac{1}{\eta}} \big(L^f_j\big)^{-\frac{1}{\eta}} \tilde{L}_j^{\frac{1}{\eta}} \tag{4c}$$

$$L^m_j : w_m = p_j Z_j * (1 - \xi_j)\big((T - s_m)H(s_m)\big)^{-\frac{1}{\eta}} \big(L^m_j\big)^{-\frac{1}{\eta}} \tilde{L}_j^{\frac{1}{\eta}} \tag{4d}$$

Combining equations 4c and 4d, we have for firm j that

$$\frac{L_j^m}{L_j^f} = \alpha_j^{-\eta} x^{\eta} \frac{(T-s_f)H(s_f)}{(T-s_m)H(s_m)}, \tag{4e}$$

where $\alpha_j := \frac{\xi}{1-\xi}$ and $x := w_f/w_m$.
Define the female wage bill in sector j as

$$I_j := \frac{w_f(T-s_f)H(s_f)L_j^f}{w_f(T-s_f)H(s_f)L_j^f + w_m(T-s_m)H(s_m)L_j^m} \tag{4f}$$

Using Equation (4e) to replace L_j^m/L_j^f in the female wage bill share, this simplifies to

$$I_j = \frac{1}{1+\alpha_j^{-\eta}x^{\eta-1}}. \tag{4g}$$

The female wage bill share in both market sectors is independent of labor units and schooling years. As in the original model by Ngai and Petrongolo (2017), the sole determinants of the female wage bill share are the wage ratio w_f/w_m and the comparative advantage of women in sector j, ξ_j. Next, total effective labor units used in the production of j are computed as a function of female effective labor units, only:

$$L_j = \left[\xi_j\big((T-s_f)H(s_f)L_j^f\big)^{\frac{\eta-1}{\eta}} + (1-\xi_j)\big((T-s_m)H(s_m)L_j^m\big)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$L_j = \left[\xi_j\big((T-s_f)H(s_f)L_j^f\big)^{\frac{\eta-1}{\eta}} + (1-\xi_j)\Big((T-s_m)H(s_m)\alpha_j^{-\eta}x^{\eta}\frac{(T-s_f)H(s_f)}{(T-s_m)H(s_m)}L_j^f\Big)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$= (T-s_f)H(s_f)L_j^f\left[\xi_j(1+\alpha_j^{-\eta}x^{\eta-1})\right]^{\frac{\eta}{\eta-1}}.$$

Combining with Equation (4g), this yields

$$L_j = \left(\frac{\xi_j}{I_j}\right)^{\frac{\eta}{\eta-1}}(T-s_f)H(s_f)L_j^f. \tag{4h}$$

Equation (4h) can then again be used in order to simplify Equations 4c-4d,

$$w_f = p_jZ_j\xi_j\big((T-s_f)H(s_f)L_j^f\big)^{-\frac{1}{\eta}}\left((T-s_f)H(s_f)L_j^f\xi_j^{\frac{\eta}{\eta-1}}I_j(x)^{\frac{\eta}{1-\eta}}\right)^{\frac{1}{\eta}}$$

$$= p_jZ_j\xi_j^{\frac{\eta}{\eta-1}}I_j(x)^{\frac{1}{1-\eta}}. \tag{4i}$$

Assuming that wages equalize across sectors (not across genders), I can combine Equation (4i) for two sectors:

$$p_i Z_i \xi_i^{\frac{\eta}{\eta-1}} I_i^{\frac{1}{1-\eta}} = w_f = p_j Z_j \xi_j^{\frac{\eta}{\eta-1}} I_j^{\frac{1}{1-\eta}}$$

s.t. the relative prices of two market commodities equal

$$\boxed{\frac{p_i}{p_j} = \frac{Z_j}{Z_i}\left(\frac{\xi_j}{\xi_i}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_i}{I_j}\right)^{\frac{1}{\eta-1}}}. \tag{4j}$$

A.3 Household Optimization

I now follow with the optimization problem for the representative couple.

$$\max_{\substack{s_m, s_f, L_l^f L_l^m \\ \{c_{jm}, L_{jh}^f, L_{jh}^m\}_{j=A,M,S}}} \int_{t=0}^{T} e^{-\rho t} dt \ \left[ln(c-\bar{c}) + \varphi ln(L_l)\right] + W(s_m) + W(s_f) \tag{5a}$$

s.t.

$$c = \left[\sum_{j=A,M,S} \omega_j c_j^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{5b}$$

$$c_j = \left[\psi_j (c_{jm})^{\frac{\sigma-1}{\sigma}} + (1-\psi_j)(c_{jh})^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{5c}$$

$$c_{jh} = Z_{jh}\left[\xi_{jh}(L_{jh}^f)^{\frac{\eta-1}{\eta}} + (1-\xi_{jh})(L_{jh}^m)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{5d}$$

$$L_l = \left[\xi_l (L_l^f)^{\frac{\eta_l-1}{\eta_l}} + (1-\xi_l)(L_l^m)^{\frac{\eta_l-1}{\eta_l}}\right]^{\frac{\eta_l}{\eta_l-1}} \tag{5e}$$

$$H = \exp\left(\frac{\zeta}{1-\nu} s_g^{1-\nu}\right), \qquad g = m, f \tag{5f}$$

$$\int_{t=0}^{T} e^{-\rho t}\left(\sum_{j=A,M,S} p_{jm} c_{jm}\right) dt = \sum_{g=m,f} \int_{t=s_g}^{T} e^{-\rho t}\left[w_g H(s_g)(L^g - L_l^g - \sum_{j=A,M,S} L_{jh}^g)\right] dt \tag{5g}$$

In what follows, I define $a_T := \int_{t=0}^{T} e^{-\rho t} dt$ and $d(s_g) := \int_{t=s_g}^{T} e^{-\rho t} dt$, $g = f, m$. The derivatives take the form:

$$c_{j_m}: \frac{\partial U}{\partial c_{j_m}} = \lambda a_T p_{j_m}, \qquad j = A, M, S \tag{6a}$$

$$L^g_{j_h}: \frac{\partial U}{\partial c_{j_h}} \frac{\partial c_{j_h}}{\partial L^g_{j_h}} = \lambda w_g d(s_g) H(s_g), \qquad j = A, M, S \tag{6b}$$

$$L^g_l: \frac{\partial U}{\partial L^g_l} = \lambda w_g d(s_g) H(s_g), \qquad g = f, m \tag{6c}$$

$$s_g: W'(s_g) = -\lambda w_g M^g \Big[d(s_g) H'(s_g) + d'(s_g) H(s_g) \Big], \qquad g = f, m \tag{6d}$$

where M^g with $g = f, m$ denotes hours in formal work of women and men ($M^g = L^g - \sum_{i=A_H, M_H, S_H, l} L^g_i$)

A.3.1 Relative prices of formal (firm commodities) vs. traditional (household commodities)

On the traditional production side for commodities $j = A, M, S$, compute the marginal rate of technical substitution by combining Equations (6b) for men and women:

$$\frac{\frac{\partial c_{j_h}}{\partial L^f_{j_h}}}{\frac{\partial c_{j_h}}{\partial L^m_{j_h}}} = \frac{w_f}{w_m}\frac{d(s_f)}{d(s_m)}\frac{H(s_f)}{H(s_m)}, \tag{7a}$$

where I define $\tilde{d} := \frac{d(s_f)}{d(s_m)}$ and $\tilde{H} := \frac{H(s_f)}{H(s_m)}$ to be the ratios of female to male lifetime in production/relative human capital levels. Since

$$\frac{\partial c_{j_h}}{\partial L^f_{j_h}} = Z_{j_h}\xi_{j_h}\left(L^f_{j_h}\right)^{-\frac{1}{\eta}} L_{j_h}^{\frac{1}{\eta}} \tag{7b}$$

$$\frac{\partial c_{j_h}}{\partial L^m_{j_h}} = Z_{j_h}(1-\xi_{j_h})\left(L^m_{j_h}\right)^{-\frac{1}{\eta}} L_{j_h}^{\frac{1}{\eta}}, \tag{7c}$$

we get male labor hours in traditional production as a function of female traditional hours (through division of the two equations):

$$\left(\frac{L^f_{j_h}}{L^m_{j_h}}\right)^{-\frac{1}{\eta}} \alpha_{j_h} = x\tilde{d}\tilde{H}$$

$$L^m_{j_h} = \alpha_{j_h}^{-\eta} x^{\eta} (\tilde{d}\tilde{H})^{\eta} L^f_{j_h} \tag{7d}$$

In this model, it will be useful to define and derive the ratio of lifetime female over total household earnings, since couples will not start their working life at the same time. I define the female lifetime earnings share as

$$I_{j_h} := \frac{w_f L^f_{j_h} d(s_f) H(s_f)}{w_f L^f_{j_h} d(s_f) H(s_f) + w_m L^m_{j_h} d(s_m) H(s_m)}. \tag{8a}$$

Using Equation (7d) to replace male by female hours in the above denominator yields

$$I_{j_h} = \frac{w_f L^f_{j_h} d(s_f) H(s_f)}{w_f L^f_{j_h} d(s_f) H(s_f)\Big(1 + \underbrace{\frac{w_m}{w_f}}_{=x^{-1}} \alpha_{j_h}^{-\eta} x^{\eta} (\tilde{d}\tilde{H})^{\eta} \underbrace{\frac{d(s_m)H(s_m)}{d(s_f)H(s_f)}}_{=(\tilde{d}\tilde{H})^{-1}}\Big)}$$

so that:

$$
\boxed{I_{j_h} = \frac{1}{1+\alpha_{j_h}^{-\eta} x^{\eta-1} \left(\tilde{d}\tilde{H}\right)^{\eta-1}}}. \tag{8b}
$$

Also, use female hours from Equation (7d) to replace $L^m_{j_h}$ in the CES for the total supply of labor in traditional production (L_{j_h}) in Equation (5d), which yields

$$
\begin{aligned}
L_{j_h} &= \left[\xi_{j_h}\left(L^f_{j_h}\right)^{\frac{\eta-1}{\eta}} + (1-\xi_{j_h})\left(\alpha_h^{-\eta} x^{\eta} (\tilde{d}\tilde{H})^{\eta} L^f_{j_h}\right)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \\
\frac{L_{j_h}}{L^f_{j_h}} &= \left[\xi_{j_h}\left(1+\alpha_{j_h}^{-\eta} (x\tilde{d}\tilde{H})^{\eta-1}\right)\right]^{\frac{\eta}{\eta-1}}
\end{aligned}
$$

$$
\boxed{\frac{L_{j_h}}{L^f_{j_h}} = \left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}}, j = A, M, S} \tag{9}
$$

Now define the implicit price of the traditional commodity: Analogue to the derivatives wrt $c_{j_m}, j = A, M, S$ in Equation (6a), I require that

$$
\frac{\partial U}{\partial c_{j_h}} = a_T p_{j_h} \lambda \tag{10}
$$

Rearrange the derivative wrt $L^f_{j_h}$ in Equation (6b) for the shadow price λ:

$$
\begin{aligned}
\lambda &= \frac{\partial U}{\partial c_{j_h}} \frac{\partial c_{j_h}}{\partial L^f_{j_h}} \frac{1}{d(s_f)H(s_f)w_f}, \text{ s.t.} \\
\frac{\partial U}{\partial c_{j_h}} \frac{1}{p_{j_h} a_T} &= \frac{\partial U}{\partial c_{j_h}} \frac{\partial c_{j_h}}{\partial L^f_{j_h}} \frac{a_T}{d(s_f)H(s_f)w_f} \\
p_{j_h} &= w_f \frac{d(s_f)H(s_f)}{a_T} \left(\frac{\partial c_{j_h}}{\partial L^f_{j_h}}\right)^{-1}
\end{aligned} \tag{11a}
$$

Replace total labor in traditional production in the derivative $\frac{\partial c_{j_h}}{\partial L^f_{j_h}}$ (see Equation (7b)) using the expression we derived for total relative to female labor in traditional production in Equation (9):

$$\frac{\partial c_{j_h}}{\partial L^f_{j_h}} = Z_{j_h}\xi_{j_h}(L^f_{j_h})^{-\frac{1}{\eta}}\left[\left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{j_h}\right]^{\frac{1}{\eta}}$$

$$\frac{\partial c_{j_h}}{\partial L^f_{j_h}} = Z_{j_h}\xi_{j_h}(L^f_{j_h})^{-\frac{1}{\eta}}(L_{j_h})^{\frac{1}{\eta}} = Z_{j_h}\xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}} \tag{11b}$$

Replace Equation (11b) in Equation (11a) to derive the implicit wage condition on the traditional production sector:

$$p_{j_h} = w_f \frac{d(s_f)H(s_f)}{a_T}\left(Z_{j_h}\xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}}\right)^{-1}$$

$$w_f = p_{j_h}Z_{j_h}\xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}}\frac{a_T}{d(s_f)H(s_f)}. \tag{11c}$$

This model assumes that wages equalize across sectors, but not across genders. Thus make use of the equation derived on the firm side relating the price of formal commodities $p_{A_F}, p_{M_F}, p_{s_m}$ to wages (see Equation (4i)), to infer relative prices of formal commodities (i_m) to the implicit price of traditional commodities (j_h):

$$p_{j_h}Z_{j_h}\xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}}\frac{a_T}{d(s_f)H(s_f)} = p_{i_m}Z_{i_m}\xi_{i_m}^{\frac{\eta}{\eta-1}}I_{i_m}^{\frac{1}{1-\eta}}$$

$$\boxed{\frac{p_{j_h}}{p_{i_m}} = \frac{Z_{i_m}}{Z_{j_h}}\left(\frac{\xi_{i_m}}{\xi_{j_h}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{j_h}}{I_{i_m}}\right)^{\frac{1}{\eta-1}}\frac{d(s_f)H(s_f)}{a_T}} \tag{12a}$$

or vice versa:

$$\boxed{\frac{p_{i_m}}{p_{j_h}} = \frac{Z_{j_h}}{Z_{i_m}}\left(\frac{\xi_{j_h}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{i_m}}{I_{j_h}}\right)^{\frac{1}{\eta-1}}\frac{a_T}{d(s_f)H(s_f)}}, \tag{12b}$$

for $i, j = A, M, S$.

Also, the relative price of two traditional commodities equals

$$p_{i_h}Z_{i_h}\xi_{i_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}}\frac{a_T}{d(s_f)H(s_f)} = p_{j_h}Z_{j_h}\xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}}\frac{a_T}{d(s_f)H(s_f)}$$

$$\frac{p_{i_h}}{p_{j_h}} = \frac{Z_{j_h}}{Z_{i_h}} \left(\frac{\xi_{j_h}}{\xi_{i_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_h}}{I_{j_h}}\right)^{\frac{1}{\eta-1}} \tag{13a}$$

for $i,j = A,M,S$. Note that human capital from s_f does not matter in Equation (13a)

A.3.2 Relative prices of traditional (HH) commodities and leisure

For the price of leisure relative to the traditional consumption good, combining the first order conditions wrt L_l^f & L_l^m, we have that

$$\frac{\frac{\partial L_l}{\partial L_l^f}}{\frac{\partial L_l}{\partial L_l^m}} = \frac{w_f d(s_f) H(s_f)}{w_m d(s_m) H(s_m)} = x\tilde{d}\tilde{H}$$

$$\frac{\xi_l \left(L_l^f\right)^{-\frac{1}{\eta_l}} L_l^{\frac{1}{\eta_l}}}{(1-\xi_l)\left(L_l^m\right)^{-\frac{1}{\eta_l}} L_l^{\frac{1}{\eta_l}}} = \frac{\xi_l}{1-\xi_l}\left(\frac{L_l^m}{L_l^f}\right)^{\frac{1}{\eta_l}} = x\tilde{d}\tilde{H}$$

$$\frac{L_l^m}{L_l^f} = \alpha_l^{-\eta_l} x^{\eta_l} (\tilde{d}\tilde{H})^{\eta_l} \tag{14}$$

Apply the same trick as before, i.e. define I_l as the female wage bill share in leisure, during working life:

$$I_l := \frac{w_f L_l^f d(s_f) H(s_f)}{w_f L_l^f d(s_f) H(s_f) + w_m L_l^m d(s_m) H(s_m)}. \tag{15a}$$

Then I_l reduces to

$$I_l = \frac{1}{1+\alpha_l^{-\eta_l} x^{\eta_l - 1} (\tilde{d}\tilde{H})^{\eta_l - 1}} \tag{15b}$$

Replace male leisure time by female leisure time in aggregate leisure L_l (see Equation (5e)) yields

$$L_l = \left[\xi_l (L_l^f)^{\frac{\eta_l-1}{\eta_l}} + (1-\xi_l)(\alpha_l^{-\eta_l} x^{\eta_l} (\tilde{d}\tilde{H})^{\eta_l} L_l^f)^{\frac{\eta_l-1}{\eta_l}}\right]^{\frac{\eta_l}{\eta_l-1}}$$

$$L_l = L_l^f \left(\frac{\xi_l}{I_l}\right)^{\frac{\eta_l}{\eta_l-1}} \tag{16}$$

Analogue to the traditional commodity, require that $\frac{\partial U}{\partial L_l^f} = \lambda a_T p_l$. Rearrange the derivative wrt L_l^f in Equation (6c) for λ and substitute:

$$\lambda = \frac{\partial U}{\partial L_l^f}\frac{1}{p_l} = \frac{a_T}{w_f d(s_f)H(s_f)}\frac{\partial U}{\partial L_l}\frac{\partial L_l}{\partial L_l^f}$$

$$p_l = w_f \frac{d(s_f)H(s_f)}{a_T}\left(\frac{\partial L_l}{\partial L_l^f}\right)^{-1} \tag{17}$$

Compute the derivative of aggregate leisure wrt female leisure time, and replace using Equation (16):

$$\frac{\partial L_l}{\partial L_l^f} = \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}} \tag{18a}$$

Then the relative price of leisure equals

$$p_l = w_f \frac{d(s_f)H(s_f)}{a_T}\left(\xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\right)^{-1} \tag{18b}$$

$$w_f = p_l \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\frac{a_T}{d(s_f)H(s_f)} \tag{18c}$$

Since wages clear across sectors, we can equate Equation (18c) and (11c) to get the price of leisure relative to the price of a traditional commodity:

$$p_l \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\frac{a_T}{d(s_f)H(s_f)} = p_{j_h} Z_{j_h} \xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}}\frac{a_T}{d(s_f)H(s_f)}$$

$$\boxed{\frac{p_l}{p_{j_h}} = Z_{j_h} \xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}} \xi_l^{\frac{\eta_l}{1-\eta_l}}\left(\frac{1}{I_l}\right)^{\frac{1}{1-\eta_l}}} \tag{19a}$$

$$\boxed{\frac{p_{j_h}}{p_l} = Z_{j_h} \xi_{j_h}^{\frac{\eta}{1-\eta}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{1-\eta}} \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}} \tag{19b}$$

And for the price ratio of leisure vs formal commodities, we have

$$p_l \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\frac{a_T}{d(s_f)H(s_f)} = p_{j_m} Z_{j_m} \xi_{j_m}^{\frac{\eta}{\eta-1}} I_{j_m}^{\frac{1}{1-\eta}}$$

$$\boxed{\frac{p_l}{p_{j_m}} = Z_{j_m} \xi_{j_m}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \left(\frac{1}{\xi_l}\right)^{\frac{\eta_l}{\eta_l-1}} I_l^{\frac{1}{\eta_l-1}} \frac{d(s_f)H(s_f)}{a_T}} \tag{19c}$$

A.3.3 Relative Expenditures

Formal (firm commodities) versus traditional (hh commodities) expenditures

Derive $MRS_{c_{jm},c_{jh}}$: We combine derivatives wrt c_{jm} and L^f_{jh} (see Equations (6a) and (6b)), to have marginal rates of substitution between formal vs traditional services:

$$MRS_{c_{jm},c_{jh}} = \frac{\frac{\partial U}{\partial c_{jm}}}{\frac{\partial U}{\partial c_{jh}}} = \frac{\lambda a_T p_{jm}}{\lambda d(s_f)H(s_f)w_f} \frac{\partial c_{jh}}{\partial L^f_{jh}} = \frac{p_{jm}}{w_f} \frac{a_T}{d(s_f)H(s_f)} \frac{\partial c_{jh}}{\partial L^f_{jh}}$$

Note that the derivative expansions needed from the chain rule on the LHS cancel out, such that

$$\frac{\frac{\partial U}{\partial c}\frac{\partial c}{\partial c_j}\frac{\partial c_j}{\partial c_{jm}}}{\frac{\partial U}{\partial c}\frac{\partial c}{\partial c_j}\frac{\partial c_j}{\partial c_{jh}}} = \frac{\frac{\partial c_j}{\partial c_{jm}}}{\frac{\partial c_j}{\partial c_{jh}}} = \frac{\psi_j c_{jm}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{(1-\psi_j) c_{jh}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}$$

$$\frac{\psi_j}{1-\psi_j}\left(\frac{c_{jm}}{c_{jh}}\right)^{-\frac{1}{\sigma}} = \frac{p_{jm}}{w_f} \frac{a_T}{d(s_f)H(s_f)} \frac{\partial c_{jh}}{\partial L^f_{jh}}$$

We know w_f from Equation (11a) that related the implicit price of traditional services to wages. Substitute and get

$$\frac{\psi_j}{1-\psi_j}\left(\frac{c_{jm}}{c_{jh}}\right)^{-\frac{1}{\sigma}} = \frac{p_{jm}}{p_{jh}} \frac{\frac{a_T}{d(s_f)H(s_f)}\frac{\partial c_{jh}}{\partial L^f_{jh}}}{\frac{a_T}{d(s_f)H(s_f)}\frac{\partial c_{jh}}{\partial L^f_{jh}}} = \frac{p_{jm}}{p_{jh}}$$

$$\left(\frac{c_{jm}}{c_{jh}}\right)^{\frac{1}{\sigma}} = \frac{p_{jh}}{p_{jm}} \frac{\psi_j}{1-\psi_j}$$

$$\boxed{\frac{c_{jm}}{c_{jh}} = \left(\frac{p_{jh}}{p_{jm}}\right)^{\sigma} \left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma}} \tag{20}$$

Relative expenditures on formal relative to traditional commodities then are:

$$\boxed{E_{jmh} = \frac{p_{jm}c_{jm}}{p_{jh}c_{jh}} = \left(\frac{p_{jh}}{p_{jm}}\right)^{\sigma-1} \left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma_j}} \tag{21a}$$

Making use of our results for relative prices p_{j_h}/p_{j_m} in Equation (12a)

$$\frac{p_{j_h}}{p_{j_m}} = \frac{Z_{j_m}}{Z_{j_h}} \left(\frac{\xi_{j_m}}{\xi_{j_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \frac{d(s_f)H(s_f)}{a_T} \tag{21b}$$

and defining $\hat{Z}_{j_{mh}} := \frac{Z_{j_m}}{Z_{j_h}} \left(\frac{\psi_j}{1-\psi_j}\right)^{\frac{\sigma}{\sigma-1}}$

$$E_{j_{mh}} = \left[\frac{Z_{j_m}}{Z_{j_h}} \left(\frac{\xi_{j_m}}{\xi_{j_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \frac{d(s_f)H(s_f)}{a_T}\right]^{\sigma-1} \left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma}$$

Eventually, the relative expenditure share of formal to traditional commodity j equals

$$\boxed{E_{j_{mh}} = \hat{Z}_{j_{mh}}^{\sigma-1} \left[\left(\frac{\xi_{j_m}}{\xi_{j_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \frac{d(s_f)H(s_f)}{a_T}\right]^{\sigma-1}}. \tag{21c}$$

Expenditures across commodity kinds

For formal commodities j_m, i_m, derive $MRS_{c_{j_m},c_{i_m}}$. Combine the derivatives wrt c_{j_m} and c_{i_m} (see Equation 6a):

$$MRS_{c_{j_m},c_{i_m}} = \frac{p_{j_m}}{p_{i_m}} = \frac{\frac{\partial U}{\partial c}\frac{\partial c}{\partial c_j}\frac{\partial c_j}{\partial c_{j_m}}}{\frac{\partial U}{\partial c}\frac{\partial c}{\partial c_i}\frac{\partial c_i}{\partial c_{i_m}}} = \frac{\omega_j}{\omega_i}\frac{c_j^{-\frac{1}{\varepsilon}}\psi_j c_{j_m}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{c_i^{-\frac{1}{\varepsilon}}\psi_i c_{i_m}^{-\frac{1}{\sigma}} c_i^{\frac{1}{\sigma}}}$$

$$\frac{p_{j_m}}{p_{i_m}} = \frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\left(\frac{c_i}{c_j}\right)^{\frac{1}{\varepsilon}}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{1}{\sigma}}\left(\frac{c_j}{c_{j_m}}\right)^{\frac{1}{\sigma}}$$

$$\left(\frac{p_{j_m}}{p_{i_m}}\right)^{\varepsilon} = \frac{c_{i_m}}{c_{j_m}}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\varepsilon-\sigma}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\sigma-\varepsilon}{\sigma}}\left(\frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\right)^{\varepsilon}$$

$$\boxed{\frac{c_{j_m}}{c_{i_m}} = \left(\frac{p_{i_m}}{p_{j_m}}\right)^{\varepsilon}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\varepsilon-\sigma}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\sigma-\varepsilon}{\sigma}}\left(\frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\right)^{\varepsilon}} \tag{22a}$$

and

$$\boxed{\frac{c_{i_m}}{c_{j_m}} = \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\varepsilon}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\sigma-\varepsilon}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\varepsilon-\sigma}{\sigma}}\left(\frac{\omega_i}{\omega_j}\frac{\psi_i}{\psi_j}\right)^{\varepsilon}} \tag{22b}$$

Problematic: The c_i, c_j in Equation (22b). Consider the inverse relative demand c_{i_m}/c_{i_h}

$$\frac{c_{i_m}}{c_{i_h}} = \left(\frac{p_{i_h}}{p_{i_m}}\right)^{\sigma}\left(\frac{\psi_i}{1-\psi_i}\right)^{\sigma}$$

in Equation (20) to derive c_i as a function of c_{i_m}:

$$c_i = \left[\psi_i c_{i_h}^{\frac{\sigma-1}{\sigma}} + (1-\psi_i)c_{i_m}^{\frac{\sigma-1}{\sigma}}\left(\frac{p_{i_m}}{p_{i_h}}\frac{1-\psi_i}{\psi_i}\right)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}}$$

$$\frac{c_i}{c_{i_m}} = \psi_i^{\frac{\sigma}{\sigma-1}}\left[1+\underbrace{\left(\frac{1-\psi_i}{\psi_i}\right)^{\sigma}\left(\frac{p_{i_m}}{p_{i_h}}\right)^{\sigma-1}}_{=E_{i_{mh}}^{-1}}\right]^{\frac{\sigma}{\sigma-1}} \tag{23a}$$

Note that the inner part of Equation (23a) equals the inverse of $E_{i_{mh}}$ in Equation (21a), so that one can write

$$\boxed{\frac{c_i}{c_{i_m}} = \psi_i^{\frac{\sigma}{\sigma-1}}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma}{\sigma-1}}} \tag{23b}$$

This equation holds for $i = A, M, S$, i.e. we also get

$$\boxed{\frac{c_j}{c_{j_m}} = \psi_j^{\frac{\sigma}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\sigma}{\sigma-1}}} \tag{23c}$$

Plug into Equation (22a), which yields

$$\frac{c_{j_m}}{c_{i_h}} = \left(\frac{p_{i_h}}{p_{j_m}}\right)^{\varepsilon}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}}\left(\frac{\omega_i}{\omega_j}\right)^{\varepsilon}\psi_i^{\sigma\frac{\varepsilon-1}{1-\sigma}}\psi_j^{\sigma\frac{1-\varepsilon}{1-\sigma}} \tag{24}$$

and since it will be useful to have everything in terms of relative expenditures, again write

$$\frac{p_{j_m}c_{j_m}}{p_{i_m}c_{i_m}} = E_{j_m i_m} = \left(\frac{p_{i_m}}{p_{j_m}}\right)^{\varepsilon-1}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}}\left(\frac{\omega_i}{\omega_j}\right)^{\varepsilon}\psi_i^{\sigma\frac{\varepsilon-1}{1-\sigma}}\psi_j^{\sigma\frac{1-\varepsilon}{1-\sigma}} \tag{25a}$$

Again, make use of the relative price equation from the formal side (see Equation (4j)) to substitute for p_s/p_g and define

$$\hat{Z}_{j_m i_m} := \frac{Z_{j_m}}{Z_{i_m}}\left(\frac{\omega_i}{\omega_j}\right)^{\frac{\varepsilon}{\varepsilon-1}}\psi_i^{\frac{\sigma}{1-\sigma}}\psi_j^{\frac{\sigma}{\sigma-1}} \tag{25b}$$

to get

$$E_{j_m i_m} = \hat{Z}_{j_m i_m}^{\varepsilon-1}\left[\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{j_m}}{I_{i_m}}\right)^{\frac{1}{1-\eta}}\right]^{\varepsilon-1}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}} \tag{25c}$$

Leisure:

$MRS_{L_l,c_{j_m}}$. Equate marginal utilities, set equal to price ratio. Note that

$$\frac{\partial U}{\partial L_l} = \lambda w_f \frac{d(s_f)H(s_f)}{a_T}\left(\frac{\partial L_l}{\partial L_{fl}}\right)^{-1} = \lambda p_l$$

by our definition of the implicit leisure price (see Equation (17)). Thus

$$\frac{\frac{\partial U}{\partial c_{jm}}}{\frac{\partial U}{\partial L_l}} = \frac{p_{jm}}{p_l}$$

$$\frac{\frac{1}{c-\bar{c}}\omega_j c_j^{-\frac{1}{\varepsilon}} c^{\frac{1}{\varepsilon}} \psi_j c_{j_m}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{\frac{\varphi}{L_l}} = \frac{p_{jm}}{p_l}$$

$$\frac{1}{\varphi}\frac{c}{c-\bar{c}}\omega_j \psi_j \left(\frac{c}{c_j}\right)^{\frac{1-\varepsilon}{\varepsilon}}\left(\frac{c_j}{c_{jm}}\right)^{\frac{1-\sigma}{\sigma}} = \frac{p_{jm}c_{jm}}{p_l L_l} \tag{26}$$

We know the relations c_j/c_{j_m} and c_i/c_{i_m} from Equation (23c) and Equation (23b)

$$\frac{c_j}{c_{jm}} = \psi_j^{\frac{\sigma}{\sigma-1}}\left(1+\frac{1}{E_{j_m j_h}}\right)^{\frac{\sigma}{\sigma-1}}$$

Also,

$$c = \left[\sum_{i=A,M,S} \omega_i (c_i)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$c = \left[\omega_j (c_j)^{\frac{\varepsilon-1}{\varepsilon}} \sum_{i=A,M,S} \frac{\omega_i}{\omega_j}\left(\frac{c_i}{c_j}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, j = A,M,S$$

$$\frac{c}{c_j} = \left[\omega_j \sum_{i=A,M,S} \frac{\omega_i}{\omega_j}\left(\frac{c_i}{c_{i_m}}\frac{c_{i_m}}{c_{jm}}\frac{c_{jm}}{c_j}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{27a}$$

Substitute for the expression containing multiple relative demands, using Equation (22b):

$$\frac{c_{i_m}}{c_{j_m}}\frac{c_i}{c_{i_m}}\frac{c_{j_m}}{c_j} = \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\varepsilon}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\sigma-\varepsilon}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\varepsilon-\sigma}{\sigma}}\left(\frac{\omega_i}{\omega_j}\frac{\psi_i}{\psi_j}\right)^{\varepsilon}\frac{c_i}{c_{i_m}}\frac{c_{j_m}}{c_j}$$
$$= \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\varepsilon}\left(\frac{c_i}{c_{i_m}}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{\omega_i}{\omega_j}\frac{\psi_i}{\psi_j}\right)^{\varepsilon}$$
$$\frac{c_i}{c_j} = \left[\left(\frac{Z_{j_m}}{Z_{i_m}}\psi_i^{\frac{\sigma}{1-\sigma}}\psi_j^{\frac{\sigma}{\sigma-1}}\right)^{\varepsilon}\right]^{-1}\left(\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{i_m}}{I_{j_m}}\right)^{\frac{1}{\eta-1}}\right)^{-\varepsilon}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\varepsilon}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon}{1-\sigma}}\left(\frac{\omega_i}{\omega_j}\right)^{\varepsilon} \tag{27b}$$

Remember that we had defined in Equation (25c)

$$E_{j_m i_m} = \hat{Z}_{j_m i_m}^{\varepsilon-1}\left[\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{j_m}}{I_{i_m}}\right)^{\frac{1}{1-\eta}}\right]^{\varepsilon-1}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}}$$
$$\text{with } \hat{Z}_{j_m i_m} := \frac{Z_{j_m}}{Z_{i_m}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}}\psi_i^{\frac{\sigma}{1-\sigma}}\psi_j^{\frac{\sigma}{\sigma-1}}$$

Thus, Equation (27b) simplifies to

$$\frac{c_i}{c_j} = \Bigg[\underbrace{\left(\hat{Z}_{j_m i_m}\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{i_m}}{I_{j_m}}\right)^{\frac{1}{\eta-1}}\right)^{\varepsilon}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}\frac{\varepsilon}{\varepsilon-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}\frac{\varepsilon}{\varepsilon-1}}}_{=E_{j_m i_m}^{\frac{\varepsilon}{\varepsilon-1}}}\Bigg]^{-1}$$
$$\times\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\varepsilon}{\varepsilon-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{27c}$$

$$\boxed{\frac{c_i}{c_j} = \left(E_{j_m i_m}\frac{E_{i_{mh}}}{E_{j_{mh}}}\frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}}} \tag{27d}$$

We then have that

$$
\begin{aligned}
\frac{c}{c_j} &= \left[\omega_j \sum_{i=A,M,S} \frac{\omega_i}{\omega_j}\left(\frac{c_i}{c_j}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
\frac{c}{c_j} &= \omega_j^{\frac{\varepsilon}{\varepsilon-1}}\left[\sum_{i=A,M,S} \frac{\omega_i}{\omega_j}\left(\left(E_{j_m i_m}\frac{E_{i_{mh}}}{E_{j_{mh}}}\frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
\frac{c}{c_j} &= \omega_j^{\frac{\varepsilon}{\varepsilon-1}}\left[\sum_{i=A,M,S}\left(E_{j_m i_m}\frac{E_{i_{mh}}}{E_{j_{mh}}}\frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{-1}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
c &= c_j \omega_j^{\frac{\varepsilon}{\varepsilon-1}} E_j^{\frac{\varepsilon}{\varepsilon-1}} \qquad (27e)
\end{aligned}
$$

where $E_j := \frac{E_{j_{mh}}}{1+E_{j_{mh}}}\sum_{i=A,M,S}\left(E_{i_m j_m}\frac{1+E_{i_{mh}}}{E_{i_{mh}}}\right)$. The expression in Equation (26) then becomes

$$
\begin{aligned}
\frac{p_{j_m}c_{j_m}}{p_l L_l} &= \frac{1}{\varphi}\frac{c}{c-\bar{c}}\omega_j \psi_j\left(\omega_j^{\frac{\varepsilon}{\varepsilon-1}}E_j^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{1-\varepsilon}{\varepsilon}}\left(\frac{c_j}{c_{j_m}}\right)^{\frac{1-\sigma}{\sigma}} \\
\frac{p_{j_m}c_{j_m}}{p_l L_l} &= \frac{1}{\varphi}\frac{c}{c-\bar{c}}\psi_j E_j^{-1}\left(\frac{c_j}{c_{j_m}}\right)^{\frac{1-\sigma}{\sigma}} \\
\frac{p_{j_m}c_{j_m}}{p_l L_l} &= \frac{1}{\varphi}\frac{c}{c-\bar{c}}\psi_j E_j^{-1}\left(\psi_j^{\frac{\sigma}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\sigma}{\sigma-1}}\right)^{\frac{1-\sigma}{\sigma}}
\end{aligned}
$$

$$
E_{j_m l} = \frac{1}{\varphi}\frac{c}{c-\bar{c}}\left[1+\sum_{i\neq j}\frac{1}{E_{j_m i_m}}\frac{E_{j_{mh}}}{E_{i_{mh}}}\frac{1+E_{i_{mh}}}{1+E_{j_{mh}}}\right]^{-1}\left(1+\frac{1}{E_{j_{mh}}}\right)^{-1} \qquad (27f)
$$

$$
E_{l j_m} = \varphi\frac{c-\bar{c}}{c}\left(1+\sum_{i\neq j}\frac{1}{E_{j_m i_m}}\frac{E_{j_{mh}}}{E_{i_{mh}}}\frac{1+E_{i_{mh}}}{1+E_{j_{mh}}}\right)\left(1+\frac{1}{E_{j_{mh}}}\right) \qquad (27g)
$$

A.4 A root function for the wage gap

A.4.1 Rewriting the Time Constraint

Use market clearing conditions: Demand of formal and traditional commodities equals supply from formal & traditional production sectors

- formal sectors $j = A_m, M_m, S_m$

$$Tc_j = Z_j \tilde{L}_j, \tag{28a}$$

- traditional sector, $j = A_h, M_h, S_h$

$$c_j = Z_j \left(\frac{\xi_j}{I_j} \right)^{\frac{\eta}{\eta-1}} L_j^f$$

Time in traditional and formal production (of all generations) cannot exceed total time endowment:

$$\frac{a_T}{d(s_f)} \left(L^g_{A_m} + L^g_{M_m} + L^g_{S_m} \right) = L^g - L^g_{A_h} + L^g_{M_h} + L^g_{S_h} - L^g_l, \tag{29a}$$

where the term $\frac{a_T}{d(s_f)}$ corrects for the fact that only adults spend time in formal work (for the young, this time is "wasted").

I now rewrite this time constraint to derive the first part of a condition. To do so I use (per period) relative expenditures of two commodities $j, k = A_m, A_h, M_m, M_h, S_m, S_h, l$, which I defined as

$$E_{kj} = \frac{p_k c_k}{p_j c_j}$$

I now derive a similar condition to that in Equation (38c) by using formal clearing conditions in expressions for (per period) relative expenditures of two commodities and leisure $j, k = A_m, A_h, M_m, M_h, S_m, S_h, l$ which we defined as

$$E_{kj} = \frac{p_k c_k}{p_j c_j}$$

Formal to Formal.
Using Equation (28a), we can derive for formal sectors $j, k = A_m, M_m, S_m$ that

$$E_{kj} = \frac{p_k c_k}{p_j c_j} = \frac{p_k Z_k}{p_j Z_j} \frac{L_k \frac{1}{T}}{L_j \frac{1}{T}}.$$

From the relative price Equation (4j), we remember that we can substitute for $p_k Z_k / p_j Z_j$:

$$\frac{p_k Z_k}{p_j Z_j}\frac{L_k}{L_j} = \left(\frac{\xi_j}{\xi_k}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_k}{I_j}\right)^{\frac{1}{\eta-1}}\frac{L_k}{\tilde{L}_j} \tag{30a}$$

Extend the fraction $\frac{L_k}{L_j} = \frac{\frac{L_k}{L_k^f}L_k^f}{\frac{L_j}{L_j^f}L_j^f}$ and replace L_j, L_k using Equation (4h):

$$\frac{L_j}{L_j^f} = \left(\frac{\xi_j}{I_j}\right)^{\frac{\eta}{\eta-1}}(T-s_f)H(s_f); \tag{30b}$$

yields

$$E_{kj} = \left(\frac{\xi_j}{\xi_k}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_k}{I_j}\right)^{\frac{1}{\eta-1}}\frac{\left(\frac{\xi_k}{I_k}\right)^{\frac{\eta}{\eta-1}}(T-s_f)H(s_f)}{\left(\frac{\xi_j}{I_j}\right)^{\frac{\eta}{\eta-1}}(T-s_f)H(s_f)}\frac{L_k^f}{L_j^f} \tag{30c}$$

such that for two formal commodities, we have

$$\frac{L_k^f}{L_j^f} = E_{kj}\frac{I_k}{I_j} \tag{30d}$$

Formal to traditional.

Expenditures on commodities produced in the formal sector $j_m = \{A_m, M_m, S_m\}$ relative to traditional commodities $i_h = \{A_h, M_h, S_h\}$ equal

$$\begin{aligned} E_{j_f i_h} &= \frac{p_{j_m} c_{j_m}}{p_{i_h} c_{i_h}} \\ &= \frac{p_{j_m} Z_{j_m}}{p_{i_h} Z_{i_h}}\frac{L_{j_m}^f}{L_{i_h}^f}\left(\frac{\xi_{j_m}}{\xi_{i_h}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{i_h}}{I_{j_m}}\right)^{\frac{\eta}{\eta-1}}H(s_f)\frac{T-s_f}{T} \end{aligned}$$

Similar to our above derivation of relative formal sector expenditures, we make use of the relative price Equation (12b) to substitute $\frac{p_{j_m} Z_{j_m}}{p_{i_h} Z_{i_h}}$:

$$\begin{aligned} E_{j_m i_h} &= \left(\frac{\xi_{i_h}}{\xi_{j_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{j_m}}{I_{i_h}}\right)^{\frac{1}{\eta-1}}\frac{a_T}{d(s_f)H(s_f)}\frac{L_{j_m}^f}{L_{i_h}^f}\left(\frac{\xi_{j_m}}{\xi_{i_h}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{i_h}}{I_{j_m}}\right)^{\frac{\eta}{\eta-1}}H(s_f)\frac{T-s_f}{T} \\ E_{j_m i_h} &= \frac{I_{i_h}}{I_{j_m}}\frac{a_T}{d(s_f)}\frac{L_{j_m}^f}{L_{i_h}^f}\frac{T-s_f}{T} \end{aligned}$$

so that we have relative labor supply equaling

$$\frac{L^f_{j_m}}{L^f_{i_h}} = E_{j_m i_h} \frac{I_{j_m}}{I_{i_h}} \frac{d(s_f)}{a_T} \frac{T}{T - s_f}. \tag{31a}$$

Leisure to traditional.

Expenditures on leisure relative to the traditional produced good equal

$$\begin{aligned} E_{l i_h} &= \frac{p_l L_l}{p_{i_h} c_{i_h}} \\ &= \frac{p_l}{p_{i_h} Z_{i_h}} \frac{\left(\frac{\xi_l}{I_l}\right)^{\frac{\eta_l}{\eta_l - 1}} L^f_l}{\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{i_h}} \end{aligned}$$

Substitute for the terms $\frac{p_l}{p_{i_h} Z_{i_h}}$ using the relative price equation Equation (19a):

$$E_{l i_h} = \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \xi_l^{\frac{\eta_l}{1-\eta_l}} \left(\frac{1}{I_l}\right)^{\frac{1}{1-\eta_l}} \frac{\left(\frac{\xi_l}{I_l}\right)^{\frac{\eta_l}{\eta_l - 1}} L^f_l}{\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{i_h}} \tag{32a}$$

We thus have the relation between traditional production hours and leisure:

$$\frac{L^f_l}{L^f_{i_h}} = E_{l i_h} \frac{I_l}{I_{i_h}} \tag{32b}$$

Traditional to traditional.

For the relation of hours in the production of one traditional commodity to another traditional commodity, define

$$\begin{aligned} E_{j_h i_h} &= \frac{p_{j_h} c_{j_h}}{p_{i_h} c_{i_h}} \\ &= \frac{p_{j_h} Z_{j_h}}{p_{i_h} Z_{i_h}} \frac{\left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{j_h}}{\left(\frac{\xi_{i_h}}{I_h}\right)^{\frac{\eta}{\eta-1}} L^f_{i_h}} \\ &= \left(\frac{\xi_{i_h}}{\xi_{j_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_h}}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \frac{\left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{j_h}}{\left(\frac{\xi_{i_h}}{I_h}\right)^{\frac{\eta}{\eta-1}} L^f_{i_h}} \end{aligned} \tag{33a}$$

s.t. we also have

$$\frac{L^f_{j_h}}{L^f_{i_h}} = E_{j_h i_h}\frac{I_{j_h}}{I_{i_h}} \tag{33b}$$

Finally, we can use the expressions in (30d) and (31a) to substitute for female labor supply in the formal goods and services sector in the total time endowment Equation (29a):

$$\begin{aligned}
L^f &= \left(L^f_{A_m} + L^f_{M_m} + L^f_{S_m}\right) + L^f_{A_h} + L^f_{M_h} + L^f_{S_h} + L^f_l \\
\frac{L^f}{L^f_{i_h}} &= \left(\frac{L^f_{A_F}}{L^f_{i_h}} + \frac{L^f_{M_F}}{L^f_{i_h}} + \frac{L^f_{S_F}}{L^f_{i_h}}\right) + \frac{L^f_{A_h}}{L^f_{i_h}} + \frac{L^f_{M_h}}{L^f_{i_h}} + \frac{L^f_{S_h}}{L^f_{i_h}} + \frac{L^f_l}{L^f_{i_h}} \\
\frac{L^f}{L^f_{i_h}} &= \sum_{j=A_m,M_m,S_m} E_{ji_h}\frac{I_j}{I_{i_h}}\frac{T}{T-s_f}\frac{d(s_f)}{a_T} + \sum_{j=A_h,M_h,S_h,l} E_{ji_h}\frac{I_j}{I_{i_h}} =: R
\end{aligned} \tag{34a}$$

Such that eventually I get

$$\boxed{\frac{L^f_{i_h}}{L^f} = \frac{1}{\sum_{j=A_m,M_m,S_m} E_{j_m i_h}\frac{I_{jm}}{I_{i_h}}\frac{d(s_f)}{a_T} + \sum_{j=A_h,M_h,S_h,l} E_{ji_h}\frac{I_j}{I_{i_h}}}}. \tag{34b}$$

For comparability of male and female hours, I set $L_m = L_f$, and require that the male time constraint holds as well:

$$\begin{aligned}
\sum_{i=A_h,M_h,S_h,l} L^m_i + \sum_{j=A_m,M_m,S_m} L^m_j &= L^m \\
\sum_{i=A_h,M_h,S_h,l} \frac{L^f_i}{L^f_{S_h}}(x\tilde{d}\tilde{H})^\eta \alpha_i^{-\eta} + \sum_{j=A_m,M_m,S_m} \frac{L^f_j}{L^f_{S_h}} x^\eta \alpha^{-\eta}\frac{T-s_f}{T-s_m}\frac{H(s_f)}{H(s_m)} &= \frac{L^m}{L^f_{S_h}} \\
\sum_{i=A_h,M_h,S_h,l} R_{iS_h}(x\tilde{d}\tilde{H})^\eta \alpha_i^{-\eta} + \sum_{j=A_m,M_m,S_m} R_{jS_h} x^\eta \alpha_j^{-\eta}\frac{T-s_f}{T-s_m}\frac{H(s_f)}{H(s_m)} &= \frac{L^m}{L^f_{S_h}}
\end{aligned} \tag{35a}$$

Defining the left-hand side as R_m, and making use of Eq. (7d) and Eq. (4e) I then get that male hours equal

$$L^m_i = \frac{L_m}{R_m}\alpha_i^{-\eta}\left(x\frac{d(s_f)H(s_f)}{d(s_m)H(s_m)}\right)^\eta \frac{L^f_i}{L^f_{S_h}}, \qquad i = A_h, M_h, S_h, l \tag{35b}$$

and

$$L_j^m = \frac{L_m}{R_m} \alpha_i^{-\eta} x^\eta \left(\frac{(T - s_f)H(s_f)}{(T - s_m)H(s_m)} \right) \frac{L_j^f}{L_{S_h}^f}, \qquad j = A_m, M_m, S_m \tag{35c}$$

A.4.2 Rewriting the Budget Constraint

Putting it all together: Rewrite the BC using the implicit prices for traditional commodities:

$$\begin{aligned} \int_{t=0}^{T} e^{-\rho t} \Big(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j \Big) dt &= \sum_{g=m,f} \int_{t=s_g}^{T} e^{-\rho t} \Big[w_g H(s_g) (L^g - \sum_{j=A,M,S,l} L_{j_h}^g) \Big] dt \\ a_T \Big(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j \Big) &= \sum_{g=m,f} d(s_g) \Big[w_g H(s_g) (L^g - \sum_{j=A,M,S,l} L_{j_h}^g) \Big] \end{aligned}$$

Take time in traditional production and leisure to the LHS:

$$a_T \left(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j + \sum_{g=m,f} w_g \frac{d(s_g)H(s_g)}{a_T} \sum_{i=A_H,M_H,S_H,l} L_i^g \right) = \sum_{g=m,f} w_g d(s_g) H(s_g) L^g \tag{36a}$$

and replace male by female leisure/time in traditional production.Then the LHS takes the form:

$$\begin{aligned} & a_T \left(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j + w_m \frac{d(s_m)H(s_m)}{a_T} \sum_{\substack{i=A_h,M_h\\ S_h,l}} L_i^m + w_f \frac{d(s_f)H(s_f)}{a_T} \sum_{\substack{i=A_h,M_h\\ S_h,l}} L_i^f \right) \\ =\, & a_T \left(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j + w_f \frac{d(s_f)H(s_f)}{a_T} \sum_{\substack{i=A_h,M_h\\ S_h,l}} L_i^f \underbrace{\left[1 + \alpha_i^{-\eta_i} (x d \tilde{H})^{\eta_i - 1} \right]}_{\text{see Equations 8b, 15b}} \right) \end{aligned} \tag{36b}$$

such that

$$a_T \left(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j + w_f \frac{d(s_f)H(s_f)}{a_T} \sum_{\substack{i=A_h,M_h\\ S_h,l}} L_i^f I_i^{-1} \right) = \sum_{g=m,f} w_g d(s_g) H(s_g) L^g \tag{36c}$$

Now remember the definition of implicit prices, given by Equations (11a) & (17):

$$p_{j_h} = w_f \frac{d(s_f)H(s_f)}{a_T} \left(\frac{\partial c_{j_h}}{\partial L_{j_h}^f} \right)^{-1}, j = A, M, S$$

Substitute into the BC:

$$a_T\left(\sum_{\substack{j=A_m,\\M_m,S_m}} p_jc_j+\sum_{\substack{i=A_h,M_h\\S_h,l}} p_iL_i^fI_i^{-1}\frac{\partial c_i}{\partial L_i^f}\right)=\sum_{g=m,f} w_gd(s_g)H(s_g)L^g \tag{36d}$$

Make use of the partial derivatives (see Equations (11b), (18a) and Equations (9), (16) :

$$a_T\left(\sum_{\substack{j=A_m,\\M_m,S_m}} p_jc_j+\sum_{\substack{i=A_h,M_h\\S_h,l}} p_iA_iL_i^f\left(\frac{\xi_i}{I_i}\right)^{\frac{\eta_i}{\eta_i-1}}\right)=\sum_{g=m,f} w_gd(s_g)H(s_g)L^g \tag{36e}$$

s.t. we have a rewritten lifetime budget constraint of the form

$$a_T\left(\sum_{j=A,M,S}\sum_{k=m,n} p_{j_k}c_{j_k}+p_lL_l\right)=\sum_{g=m,f} w_gd(s_g)H(s_g)L^g \tag{36f}$$

The idea now is to derive a condition including expenditure shares relative to traditional services $j_h=S_H$

$$a_T\left(\sum_{j=A,M,S}\sum_{k=m,n}\frac{p_{j_k}c_{j_k}}{p_{S_h}c_{S_h}}+\frac{p_lL_l}{p_{S_h}c_{S_h}}\right)=\sum_{g=m,f}\frac{w_gd(s_g)H(s_g)L^g}{p_{S_h}c_{S_h}}$$

$$\frac{1}{a_T\left(\sum_{j=A,M,S}\sum_{k=m,n}E_{i_kS_h}+E_{lS_h}\right)}w_fd(s_f)H(s_f)L^f=p_{S_h}c_{S_h}\frac{w_fd(s_f)H(s_f)L^f}{\sum_{g=m,f}w_gd(s_g)H(s_g)L^g} \tag{36g}$$

Now note that the RHS of Equation (36g) contains the expression for the ratio of female in total HH lifetime earnings. We can rewrite this part as

$$I_L=\frac{w_fL^fd(s_f)H(s_f)}{L^fw_fH(s_f)d(s_f)+w_mL^md(s_m)H(s_m)}$$
$$I_L=\frac{1}{1+\left(\tilde{L}x\tilde{d}\tilde{H}\right)^{-1}} \tag{37}$$

We make use of Equation (37) and use the implicit definition of p_{S_h}, derived in Equation (11a), which we use to replace p_{S_h} in Equation (36g)

$$\frac{1}{a_T\left(\sum_{j=A,M,S}\sum_{k=m,n}E_{i_kS_h}+E_{lS_h}\right)}w_fd(s_f)H(s_f)L^f=p_{S_h}c_{S_h}I_L$$

$$\frac{1}{\sum_{j=A,M,S}\sum_{k=m,n}E_{i_kS_h}+E_{lS_h}}L^f=\left(\frac{\partial c_{S_h}}{\partial L_{S_h}^f}\right)^{-1}c_{S_h}I_L \tag{38a}$$

and using Equations (11b) as well as (9) to replace L_{S_h} in $c_{S_h} = A_{S_h} L_{S_h}$

$$\frac{1}{\sum\limits_{j=A,M,S} \sum\limits_{k=m,n} E_{i_k S_h} + E_{l S_h}} L^f = \left[A_{S_h} \xi_{S_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{S_h}} \right)^{\frac{1}{\eta-1}} \right]^{-1} c_{S_h} I_L \tag{38b}$$

Simplifiying this expression yields

$$\boxed{\frac{L^f_{S_h}}{L^f} = \frac{1}{I_L \sum\limits_{\substack{j=A_m,M_m,S_m \\ A_h,M_h,S_h,l}} \frac{E_{jS_h}}{I_{S_h}}}} \tag{38c}$$

I combine this with Equation (34b) to arrive at a condition that the wage ratio x and years of schooling s_f, S_F must satisfy:

$$\boxed{\frac{1}{I_L \sum\limits_{\substack{j=A_m,M_m,S_m \\ A_h,M_h,S_h,l}} \frac{E_{jS_h}}{I_{S_h}}} - \frac{1}{\sum\limits_{\substack{j=A_m, \\ M_m,S_m}} E_{j_m S_h} \frac{I_{jm}}{I_{S_h}} \frac{d(s_f)}{a_T} + \sum\limits_{\substack{j=A_h, \\ M_h,S_h,l}} E_{jS_h} \frac{I_j}{I_{S_h}}} = 0.} \tag{39}$$

A.5 Deriving the conditions for schooling

We can use the condition given by the first order conditions wrt female schooling and female hours in non-market production, Equations (6b) and (5d) as a starting point for the derivation of a condition on the optimal level of (female) schooling years:

$$\beta_f s_f = -M^f \left[\frac{H'(s_f)}{H(s_f)} + \frac{d'(s_f)}{d(s_f)}\right] \frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^f_{i_h}} a_T \tag{40a}$$

Focus on the second part of the rhs. We have that

$$\frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^f_{i_h}} = \omega_i \left(\frac{c}{c_i}\right)^{\frac{1}{\varepsilon}} (1-\psi_i) \left(\frac{c_i}{c_{i_h}}\right)^{\frac{1}{\sigma}} A_{i_h} \xi_{i_h} (L^f_{i_h})^{-\frac{1}{\eta}} L_h^{\frac{1}{\eta}}$$

Since $L^f_{i_h} = c_{i_h}/A_{i_h}$,

$$\begin{aligned}
\frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^f_{i_h}} &= \frac{1}{c-\bar{c}} \omega_i \left(\frac{c}{c_i}\right)^{\frac{1}{\varepsilon}} (1-\psi_i) \left(\frac{c_i}{c_{i_h}}\right)^{\frac{1}{\sigma}} A_{i_h} \xi_{i_h} (L^f_{i_h})^{-\frac{1}{\eta}} \left(\frac{c_{i_h}}{A_{i_h}}\right)^{\frac{1}{\eta}} \\
&= \frac{c}{c-\bar{c}} c^{\frac{1-\varepsilon}{\varepsilon}} c_i^{\frac{\varepsilon-\sigma}{\varepsilon\sigma}} c_{i_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}} \omega_i (1-\psi_i) A_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h}
\end{aligned} \tag{40b}$$

Now let's replace c_i by c_{i_h}, c by c_i and c_{i_h} by $L^f_{i_h}$ one by one:

1. c_i by c_{i_h}: From Equation (20) we know relative demand $\frac{c_{im}}{c_{i_h}} = \left(\frac{p_{i_h}}{p_{i_m}}\right)^{\sigma} \left(\frac{\psi_i}{1-\psi_i}\right)^{\sigma}$, so that we can replace c_{i_m} in the CES-aggregator for total services c_i:

$$\begin{aligned}
c_i &= \left[\psi_i \left((\frac{p_{i_h}}{p_{i_m}})^{\sigma} (\frac{\psi_i}{1-\psi_i})^{\sigma} c_{i_h}\right)^{\frac{\sigma-1}{\sigma}} + (1-\psi_i) c_{i_h}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\
&= c_{i_h}\left[(1-\psi_i)\left(1 + \underbrace{(\frac{p_{i_h}}{p_{i_m}})^{\sigma-1} (\frac{\psi_i}{1-\psi_i})^{\sigma}}_{=E_{i_{mh}}}\right)\right]^{\frac{\sigma}{\sigma-1}}
\end{aligned}$$

   which we can simplify using Equation (21a) to

$$c_i = (1-\psi_i)^{\frac{\sigma}{\sigma-1}} \left[1 + E_{i_{mh}}\right]^{\frac{\sigma}{\sigma-1}} c_{i_h} \tag{40c}$$

2. c by c_i: From our computations of the marginal rates of substition, we know c_j/c_i by

Equation (27d):

$$\frac{c_j}{c_i} = \left(E_{j_m i_h} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Thus, aggregate consumption c relative to c_i equals

$$\frac{c}{c_i} = \left[\omega_i \sum_{j=A,M,S} \frac{\omega_j}{\omega_i} \left(\frac{c_j}{c_i}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\frac{c}{c_i} = \omega_i^{\frac{\varepsilon}{\varepsilon-1}} \left[\sum_{j=A,M,S} \frac{\omega_j}{\omega_i} \left(\left(E_{j_m i_m} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\frac{c}{c_i} = \omega_i^{\frac{\varepsilon}{\varepsilon-1}} \left[\sum_{j=A,M,S} E_{j_m i_m} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

Define

$$E_i := \frac{E_{i_{mh}}}{1+E_{i_{mh}}} \sum_{j=A,M,S} E_{j_m i_m} \frac{1+E_{j_{mh}}}{E_{j_{mh}}} \tag{40d}$$

such that

$$\boxed{c = c_i \omega_i^{\frac{\varepsilon}{\varepsilon-1}} E_i^{\frac{\varepsilon}{\varepsilon-1}}} \tag{40e}$$

3. Remember that $c_{i_h} = A_{i_h} L_{i_h}$, where total relative to female hours in production of c_{i_h} (see Equation (9)) implies that

$$c_{i_h} = A_{i_h} \left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} L_{i_h}^f \tag{40f}$$

Now make use of Equation (40e) to replace c using c_i in Equation (40b):

$$\frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L_{i_h}^f} = \frac{c}{c-\bar{c}} \left(c_i \omega_i^{\frac{\varepsilon}{\varepsilon-1}} E_i^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{1-\varepsilon}{\varepsilon}} c_i^{\frac{\varepsilon-\sigma}{\varepsilon\sigma}} c_{i_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{L_{i_h}^f}\right)^{\frac{1}{\eta}} \omega_i (1-\psi_i) A_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h}$$

$$= \frac{c}{c-\bar{c}} E_i^{-1} c_i^{\frac{1-\sigma}{\sigma}} c_{i_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{L_{i_h}^f}\right)^{\frac{1}{\eta}} (1-\psi_i) A_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h} \tag{40g}$$

We then use Equation (40c) to replace c_i by c_{i_h}:

$$\frac{1}{c-\bar{c}}\frac{\delta c}{\delta c_i}\frac{\delta c_i}{\delta c_{i_h}}\frac{\delta c_{i_h}}{\delta L^f_{i_h}} = \frac{c}{c-\bar{c}}E_i^{-1}\left((1-\psi_i)^{\frac{\sigma}{\sigma-1}}\left[1+E_{i_{mh}}\right]^{\frac{\sigma}{\sigma-1}}c_{i_h}\right)^{\frac{1-\sigma}{\sigma}}c_{i_h}^{\frac{\sigma-\eta}{\sigma\eta}}\left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}}(1-\psi_i)A_{i_h}^{\frac{\eta-1}{\eta}}\xi_{i_h}$$

$$= \frac{c}{c-\bar{c}}E_i^{-1}\left(1+E_{i_{mh}}\right)^{-1}c_{i_h}^{\frac{1-\eta}{\eta}}\left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}}A_{i_h}^{\frac{\eta-1}{\eta}}\xi_{i_h} \quad (40h)$$

Now substitute c_{i_h} using Equation (40f):

$$\frac{1}{c-\bar{c}}\frac{\delta c}{\delta c_i}\frac{\delta c_i}{\delta c_{i_h}}\frac{\delta c_{i_h}}{\delta L^f_{i_h}} = \frac{c}{c-\bar{c}}E_i^{-1}\left(1+E_{i_{mh}}\right)^{-1}\left(A_{i_h}\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}}L^f_{i_h}\right)^{\frac{1-\eta}{\eta}}\left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}}A_{i_h}^{\frac{\eta-1}{\eta}}\xi_{i_h}$$

$$\frac{c}{c-\bar{c}}E_i^{-1}\left(1+E_{i_{mh}}\right)^{-1}\frac{I_{i_h}}{L^f_{i_h}} \quad (40i)$$

We then have that the female schooling condition takes the form

$$W'(s_f) = -\frac{M_f}{L^f_{i_h}}\left[\frac{H'(s_f)}{H(s_f)}+\frac{d'(s_f)}{d(s_f)}\right]\frac{c}{c-\bar{c}}E_i^{-1}\left(1+E_{i_{mh}}\right)^{-1}I_{i_h}a_T \quad (40j)$$

This can be further simplified when assuming that $\bar{c}=0$, and setting $i_n = S_n$:

$$W'(s_f) = -\frac{M_f}{L^f_{S_h}}\hat{H}(s_f)E_S^{-1}\left(1+E_{S_{mh}}\right)^{-1}I_{S_h}a_T. \quad (40k)$$

Making use of Equation (31a) that $L^f_{S_h} = L^f_{S_m}\frac{1}{E_{S_{mh}}}\frac{I_{S_h}}{I_{S_m}}\frac{a_T}{d(s_f)}\frac{T-s_f}{T}$

$$W'(s_f) = -\frac{M_f}{L^f_{S_m}}\hat{H}(s_f)\left[\sum_{j=A,M,S}E_{j_m S_m}\frac{1+E_{j_{mh}}}{E_{j_{mh}}}\right]^{-1}I_{S_m}d(s_f)\frac{T}{T-s_f} \quad (40l)$$

On the male side,

$$W'(s_m) = -M^m\left[\frac{H'(s_m)}{H(s_m)}+\frac{d'(s_m)}{d(s_m)}\right]\frac{1}{c-\bar{c}}\frac{\delta c}{\delta c_i}\frac{\delta c_i}{\delta c_{i_h}}\frac{\delta c_{i_h}}{\delta L^m_{i_h}}a_T.$$

Note the derivative $\frac{\delta c_{i_h}}{\delta L^m_{i_h}}$, which takes the form

$$\begin{aligned}
\frac{\delta c_{i_h}}{\delta L^m_{i_h}} &= A_{i_h}(1-\xi_{i_h})\left(\frac{L_{i_h}}{L^m_{i_h}}\right)^{\frac{1}{\eta}} \\
&= A_{i_h}(1-\xi_{i_h})\left(\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} x^{-\eta}\left(\tilde{d}\tilde{H}\right)^{-\eta}\alpha^{\eta}_{i_h}\right)^{\frac{1}{\eta}} \text{ by Equation (7d) and Equation (9)} \\
&= \underbrace{A_{i_h}\xi_{i_h}^{\frac{\eta}{1-\eta}}\left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}}}_{\text{Known from eq. (11b)}}\left(x\tilde{H}\tilde{d}\right)^{-1} \\
&= \frac{\delta c_{i_h}}{\delta L^f_{i_h}}\left(x\tilde{H}\tilde{d}\right)^{-1} \qquad (41a)
\end{aligned}$$

The condition then becomes

$$W'(s_m) = -\frac{M_m}{L^f_{i_h}}\hat{H}(s_m)\frac{c}{c-\bar{c}}E_i^{-1}\left(1+E_{i_{mh}}\right)^{-1} I_{i_h}\left(x\tilde{d}\tilde{H}\right)^{-1} a_T \qquad (42)$$

The condition relating female to male schooling directly is derived by combining the male and female conditions:

$$\frac{W'(s_f)}{W'(s_m)} = \frac{M_f}{M_m}\frac{\hat{H}(s_f)}{\hat{H}(s_m)}x\tilde{d}\tilde{H} \qquad (43)$$

B Extended Model

B.1 Short description

This is a brief description of the model setup in which I assume that both, formal as well as traditional work take place after schooling is finished. The full setup and optimization is derived in the next section. The point of this extension is to clarify how the main result is affected if schooling comes at the opportunity cost of shortening the life span during which agents can engage in both forms of work.

To address this, I adapt the supply side of the model. Different generations now pool their hours in traditional production.[1] Formally, Eq. (1a) and (1b) become

$$c_{j_h} = Z_{j_h} L_{j_h} \tag{44a}$$

$$L_{j_h} = \left[\xi_{j_h}\left((T-s_f)l^f_{j_h}\right)^{\frac{\eta-1}{\eta}} + (1-\xi_{j_h})\left((T-s_m)l^m_{j_h}\right)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}. \tag{44b}$$

I leave the optimization details to the model appendix. The main difference to the benchmark model is that the female schooling condition (11a) now reads

$$-\beta_f e^{-\rho s_f} + \frac{a_T}{T-s_f}(\tilde{E}_A^{-1} + \tilde{E}_M^{-1} + \tilde{E}_S^{-1}) = a_T \frac{M^f}{l^f_{S_h}} \hat{H}(s_f) \tilde{E}_S^{-1}. \tag{44c}$$

For brevity, I bundle the terms on relative expenditures.[2] The second term on the left-hand-side captures an additional indirect effect that schooling has on utility through foregone consumption: holding hours constant, a marginal increase in years of schooling reduces traditional output. This is because less generations engage in traditional production.

I calibrate this model as in Section 4.1.3. But now I also recalibrate the female weights in the traditional sectors ξ_{A_h}, ξ_{M_h} and ξ_{S_h} (in the benchmark, I held them fixed for middle- and high-income countries). I do so by targeting how many hours men and women who have completed education spend in each traditional sector relative to each other. In the data, traditional hours fall for both genders, but they fall relatively more for men than for women. This is why the male/female ratio in traditional hours falls from 0.56 in low to 0.45 in high-income countries (Table 1).

Table (1) shows that in this extended setup, the predicted male/female ratio in years of schooling declines to 0.96 This is better than the benchmark (0.91), but still less than the data (1.05). Men's and women's years of schooling rise to 14.1 and 14.8 years, respectively, which is

[1] For example, think of a village in which each household owns a plot, and households go from plot to plot to harvest together and then share the produce.

[2] Note that for $i = A,M,S$, $\tilde{E}_i := E_{i_{mh}} \sum_{j=A,M,S} E_{j_m i_m} \frac{1+E_{j_{mh}}}{E_{j_{mh}}} I_{i_h}^{-1}$

Table 1: Gender gaps in formal and traditional hours as well as years of schooling.

	Country Income Group		
	Low	Middle	High
Data & Extended model			
Male/female formal hours	2.14	1.94	1.60
Male/female traditional hours	0.56	0.43	0.45
Extended model			
Male schooling years	6.0	10.7	14.1
Female schooling years	5.0	9.1	14.8
Male/female schooling years	1.20	1.18	0.96

Notes: Male/female ratios in formal and traditional hours, as targeted from the data. Men's and women's years of schooling and the schooling year ratio, as predicted in the extended model where agents can engage in traditional work only after schooling is completed.

even more pronounced than in the benchmark model (12 and 13.2). The reason is the following: as traditional hours decline across country income groups, a marginal rise in years of schooling comes at a lower indirect utility cost (from reduced consumption of traditional commodities). Therefore, years of schooling rise even faster than in the benchmark model, which does not incorporate this effect.

B.2 Firm Optimization

I begin with the derivation of the solution of a representative firm j's ($j = A_m, M_m, S_m$) maximization problem.

$$\max_{L_j^f, L_j^m} p_j Z_j L_j - w_m(T - s_m)H(s_m)L_j^m - w_f(T - s_f)H(s_f)L_j^f \tag{45a}$$

s.t.

$$L_j = \left[\xi_j\big(H(s_f)(T - s_f)L_j^f\big)^{\frac{\eta-1}{\eta}} + (1 - \xi_j)\big(H(s_m)(T - s_m)L_j^m\big)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{45b}$$

$$\tag{45c}$$

First order conditions with respect to L_j^f and L_j^m yield

$$L_j^f : w_f = p_j Z_j * \xi_j\big((T - s_f)H(s_f)\big)^{-\frac{1}{\eta}}\big(L_j^f\big)^{-\frac{1}{\eta}}\tilde{L}_j^{\frac{1}{\eta}} \tag{45d}$$

$$L_j^m : w_m = p_j Z_j * (1 - \xi_j)\big((T - s_m)H(s_m)\big)^{-\frac{1}{\eta}}\big(L_j^m\big)^{-\frac{1}{\eta}}\tilde{L}_j^{\frac{1}{\eta}} \tag{45e}$$

Combining equations 45d and 45e, we have for firm j that

$$\frac{L_j^m}{L_j^f} = \alpha_j^{-\eta} x^{\eta} \frac{(T - s_f)H(s_f)}{(T - s_m)H(s_m)}, \tag{45f}$$

where $\alpha_j := \frac{\xi}{1-\xi}$ and $x := w_f/w_m$.
Define the female wage bill in sector j as

$$I_j := \frac{w_f(T - s_f)H(s_f)L_j^f}{w_f(T - s_f)H(s_f)L_j^f + w_m(T - s_m)H(s_m)L_j^m} \tag{45g}$$

Using Equation (45f) to replace L_j^m/L_j^f in the female wage bill share, this simplifies to

$$I_j = \frac{1}{1 + \alpha_j^{-\eta} x^{\eta-1}}. \tag{45h}$$

The female wage bill share in both formal sectors is independent of labor units and schooling years. As in the original model by Ngai and Petrongolo (2017), the sole determinants of the female wage bill share are the wage ratio w_f/w_m and the comparative advantage of women in sector j, ξ_j. Next, total effective labor units used in the production of j are computed as a function of female effective labor units, only:

$$L_j = \left[\xi_j \left((T - s_f) H(s_f) L_j^f \right)^{\frac{\eta - 1}{\eta}} + (1 - \xi_j) \left((T - s_m) H(s_m) L_j^m \right)^{\frac{\eta - 1}{\eta}} \right]^{\frac{\eta}{\eta - 1}}$$

$$L_j = \left[\xi_j \left((T - s_f) H(s_f) L_j^f \right)^{\frac{\eta - 1}{\eta}} + (1 - \xi_j) \left((T - s_m) H(s_m) \alpha_j^{-\eta} x^{\eta} \frac{(T - s_f) H(s_f)}{(T - s_m) H(s_m)} L_j^f \right)^{\frac{\eta - 1}{\eta}} \right]^{\frac{\eta}{\eta - 1}}$$

$$= (T - s_f) H(s_f) L_j^f \left[\xi_j (1 + \alpha_j^{-\eta} x^{\eta - 1}) \right]^{\frac{\eta}{\eta - 1}}.$$

Combining with Equation (45h), this yields

$$L_j = \left(\frac{\xi_j}{I_j} \right)^{\frac{\eta}{\eta - 1}} (T - s_f) H(s_f) L_j^f. \tag{45i}$$

Equation (45i) can then again be used in order to simplify Equations 4c-4d,

$$w_f = p_j Z_j \xi_j \left((T - s_f) H(s_f) L_j^f \right)^{-\frac{1}{\eta}} \left((T - s_f) H(s_f) L_j^f \xi_j^{\frac{\eta}{\eta - 1}} I_j^{\frac{\eta}{1 - \eta}} \right)^{\frac{1}{\eta}}$$

$$= p_j Z_j \xi_j^{\frac{\eta}{\eta - 1}} I_j^{\frac{1}{1 - \eta}}. \tag{45j}$$

Assuming that wages equalize across sectors (not across genders), I can combine Equation (4i) for two sectors:

$$p_i Z_i \xi_i^{\frac{\eta}{\eta - 1}} I_i^{\frac{1}{1 - \eta}} = w_f = p_j Z_j \xi_j^{\frac{\eta}{\eta - 1}} I_j^{\frac{1}{1 - \eta}}$$

s.t. the relative prices of two firm commodities equal

$$\boxed{\frac{p_i}{p_j} = \frac{Z_j}{Z_i} \left(\frac{\xi_j}{\xi_i} \right)^{\frac{\eta}{\eta - 1}} \left(\frac{I_i}{I_j} \right)^{\frac{1}{\eta - 1}}}. \tag{45k}$$

B.3 Household Optimization

I now follow with the optimization problem for the representative couple. For simplicity, I drop the subscript in the setup of the maximization problem.

$$\max_{\substack{s_m, s_f, L_l^f L_l^m \\ \{c_{j_m}, L_{j_h}^f, L_{j_h}^m\}_{j=A,M,S}}} \int_{t=0}^{T} e^{-\rho t} dt \quad \Big[ln(c - \bar{c}) + \varphi ln(L_l) \Big] + W(s_m) + W(s_f) \tag{46a}$$

s.t.

$$c = \Big[\sum_{j=a,m,s} \omega_j c_j^{\frac{\varepsilon-1}{\varepsilon}} \Big]^{\frac{\varepsilon}{\varepsilon-1}} \tag{46b}$$

$$c_j = \Big[\psi_j (c_{j_m})^{\frac{\sigma-1}{\sigma}} + (1-\psi_j)(c_{j_h})^{\frac{\sigma-1}{\sigma}} \Big]^{\frac{\sigma}{\sigma-1}} \tag{46c}$$

$$c_{j_h} = \frac{1}{T} Z_{j_h} \Big[\xi_j \Big((T - s_f) L_{j_h}^f \Big)^{\frac{\eta-1}{\eta}} + (1-\xi_j) \Big((T - s_m) L_{j_h}^m \Big)^{\frac{\eta-1}{\eta}} \Big]^{\frac{\eta}{\eta-1}} \tag{46d}$$

$$L_l = \Big[\xi_l (L_l^f)^{\frac{\eta_l-1}{\eta_l}} + (1-\xi_l)(L_l^m)^{\frac{\eta_l-1}{\eta_l}} \Big]^{\frac{\eta_l}{\eta_l-1}} \tag{46e}$$

$$H = \exp\Big(\frac{\zeta}{1-\nu} s_g^{1-\nu} \Big), \qquad g = m, f \tag{46f}$$

$$\int_{t=0}^{T} e^{-\rho t} \Big(\sum_{j=A,M,S} p_j c_{j_m} \Big) dt = \sum_{g=m,f} \int_{t=s_g}^{T} e^{-\rho t} \Big[w_g H(s_g)(L^g - L_l^g - \sum_{j=A,M,S} L_{j_h}^g) \Big] dt \tag{46g}$$

In what follows, I define $a_T := \int_{t=0}^{T} e^{-\rho t} dt$ and $d(s_g) := \int_{t=s_g}^{T} e^{-\rho t} dt$, $g = f, m$. The derivatives take the form:

$$c_{j_m} : a_T \frac{\delta U}{\delta c_{j_m}} = \lambda a_T p_{j_m}, \qquad j = A, M, S \tag{47a}$$

$$L_{j_h}^g : a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta L_{j_h}^g} = \lambda w_g d(s_g) H(s_g), \qquad j = A, M, S \tag{47b}$$

$$L_l^g : a_T \frac{\delta U}{\delta L_l^g} = \lambda w_g d(s_g) H(s_g), \qquad g = f, m \tag{47c}$$

$$s_g : W'(s_g) + \sum_{j=A_h, M_h, S_h} a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_g} = -\lambda w_g M^g \Big[d(s_g) H'(s_g) + d'(s_g) H(s_g) \Big] \qquad g = f, m \tag{47d}$$

where M^g for $g = f,m$ denotes formal hours of women and men ($M^g = L^g - \sum_{i=A_h,M_h,S_h,l} L_i^g$)

B.3.1 Firm vs. household commodity prices

On the traditional production side for commodities $j = A,M,S$, compute the marginal rate of technical substitution by combining Equations (47b) for both genders:

$$\frac{\frac{\delta c_{j_h}}{\delta L_{j_h}^f}}{\frac{\delta c_{j_h}}{\delta L_{j_h}^m}} = \frac{w_f}{w_m}\frac{d(s_f)}{d(s_m)}\frac{H(s_f)}{H(s_m)}, \tag{48a}$$

where I define $\tilde{d} := \frac{d(s_f)}{d(s_m)}$ and $\tilde{H} := \frac{H(s_f)}{H(s_m)}$ to be the ratios of female to male lifetime in production/relative human capital levels. Since

$$\frac{\delta c_{j_h}}{\delta L_{j_h}^f} = \frac{1}{T} Z_{j_h}\xi_{j_h}(T - s_f)^{\frac{\eta-1}{\eta}}\left(L_{j_h}^f\right)^{-\frac{1}{\eta}} L_{j_h}^{\frac{1}{\eta}} \tag{48b}$$

$$\frac{\delta c_{j_h}}{\delta L_{j_h}^m} = \frac{1}{T} Z_{j_h}(1-\xi_{j_h})(T - s_m)^{\frac{\eta-1}{\eta}}\left(L_{j_h}^m\right)^{-\frac{1}{\eta}} L_{j_h}^{\frac{1}{\eta}}, \tag{48c}$$

we get male labor hours in traditional production as a function of female traditional hours (through division of the two equations):

$$\left(\frac{L_{j_h}^f}{L_{j_h}^m}\right)^{-\frac{1}{\eta}} \alpha_{j_h}(\tilde{T})^{\frac{\eta-1}{\eta}} = x\tilde{d}\tilde{H}$$

$$L_{j_h}^m = \alpha_{j_h}^{-\eta} x^{\eta}(\tilde{d}\tilde{H})^{\eta}\tilde{T}^{1-\eta} L_{j_h}^f \tag{48d}$$

In this model, it will be useful to define and derive the ratio of lifetime female over total household earnings, since couples will not start their working life at the same time. I define the female lifetime earnings share as

$$I_{j_h} := \frac{w_f L_{j_h}^f d(s_f) H(s_f)}{w_f L_{j_h}^f d(s_f) H(s_f) + w_m L_{j_h}^m d(s_m) H(s_m)}. \tag{49a}$$

Using Equation (48d) to replace male by female hours in the above denominator yields

$$I_{j_h} := \frac{1}{1 + (x\tilde{d}\tilde{H})^{-1}\alpha_{j_h}^{-\eta} x^{\eta}(\tilde{d}\tilde{H})^{\eta}\tilde{T}^{1-\eta}}, \tag{49b}$$

So that

$$\boxed{I_{j_h} = \frac{1}{1+\alpha_{j_h}^{-\eta}\left(xd\tilde{H}\right)^{\eta-1}\tilde{T}^{1-\eta}}}. \tag{49c}$$

Also, use female hours from Equation (48d) to replace $L_{j_h}^m$ in the CES for the total supply of labor in traditional production (L_{j_h}) in Equation (46d), which yields

$$L_{j_h} = \left[\xi_{j_h}\left((T-s_f)L_{j_h}^f\right)^{\frac{\eta-1}{\eta}} + (1-\xi_{j_h})\left((T-s_m)L_{j_h}^m\right)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$L_{j_h} = (T-s_f)L_{j_h}^f\left[\xi_{j_h}\left(1+\alpha_{j_h}^{-\eta}(xd\tilde{H})^{\eta-1}\tilde{T}^{1-\eta}\right]^{\frac{\eta}{\eta-1}}$$

$$\boxed{\frac{L_{j_h}}{L_{j_h}^f} = \left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}}(T-s_f), j=A,M,S} \tag{50}$$

Now define the implicit price of the traditional commodity: Analogue to the derivatives wrt $c_{j_m}, j=A,M,S$ in Equation (47a), I require that

$$\frac{\delta U}{\delta c_{j_h}} = p_{j_h}\lambda \tag{51}$$

Rearrange the derivative wrt $L_{j_h}^f$ in Equation (47b) for the shadow price λ:

$$\lambda = \frac{\delta U}{\delta c_{j_h}}\frac{\delta c_{j_h}}{\delta L_{j_h}^f}\frac{a_T}{d(s_f)H(s_f)w_f}, \text{ s.t.}$$

$$\frac{\delta U}{\delta c_{j_h}}\frac{1}{p_{j_h}} = \frac{\delta U}{\delta c_{j_h}}\frac{\delta c_{j_h}}{\delta L_{j_h}^f}\frac{a_T}{d(s_f)H(s_f)w_f}$$

$$p_{j_h} = w_f\frac{d(s_f)H(s_f)}{a_T}\left(\frac{\delta c_{j_h}}{\delta L_{j_h}^f}\right)^{-1} \tag{52a}$$

Replace total labor in traditional production in the derivative $\frac{\delta c_{j_h}}{\delta L_{j_h}^f}$ (see Equation (48b)) using the expression we derived for labor in Equation (50):

$$\begin{aligned}
\frac{\delta c_{j_h}}{\delta L^f_{j_h}} &= \frac{1}{T} Z_{j_h} \xi_{j_h} (T-s_f)^{\frac{\eta-1}{\eta}} (L^f_{j_h})^{-\frac{1}{\eta}} \left((\frac{\xi_{j_h}}{I_{j_h}})^{\frac{\eta}{\eta-1}} (T-s_f) L^f_{j_h} \right)^{\frac{1}{\eta}} \\
\frac{\delta c_{j_h}}{\delta L^f_{j_h}} &= Z_{j_h} \xi_{j_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{j_h}} \right)^{\frac{1}{\eta-1}} \frac{T-s_f}{T}
\end{aligned} \tag{53a}$$

Replace Equation (53a) in Equation (52a) to derive the implicit wage condition on the traditional production sector:

$$\begin{aligned}
p_{j_h} &= w_f \frac{d(s_f)H(s_f)}{a_T} \left(Z_{j_h} \xi_{j_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{j_h}} \right)^{\frac{1}{\eta-1}} \frac{T-s_f}{T} \right)^{-1} \\
w_f &= p_{j_h} Z_{j_h} \xi_{j_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{j_h}} \right)^{\frac{1}{\eta-1}} \frac{a_T}{d(s_f)H(s_f)} \frac{T-s_f}{T}.
\end{aligned} \tag{54}$$

This model assumes that wages equalize across sectors, but not across genders. Thus make use of the equation derived on the firm side relating the price of firm commodities $p_{A_m}, p_{M_m}, p_{S_m}$ to wages (see Equation (45j)), to infer relative prices of firm commodities (i_m) to the implicit price of traditional commodities (j_h):

$$p_{j_h} Z_{j_h} \xi_{j_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{j_h}} \right)^{\frac{1}{\eta-1}} \frac{a_T}{d(s_f)H(s_f)} \frac{T-s_f}{T} = p_{i_m} Z_{i_m} \xi_{i_m}^{\frac{\eta}{\eta-1}} I_{i_m}^{\frac{1}{1-\eta}}$$

$$\boxed{\frac{p_{j_h}}{p_{i_m}} = \frac{Z_{i_m}}{Z_{j_h}} \left(\frac{\xi_{i_m}}{\xi_{j_h}} \right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_h}}{I_{i_m}} \right)^{\frac{1}{\eta-1}} \frac{d(s_f)H(s_f)}{a_T} \frac{T}{T-s_f}} \tag{55a}$$

or vice versa:

$$\boxed{\frac{p_{i_m}}{p_{j_h}} = \frac{Z_{j_h}}{Z_{i_m}} \left(\frac{\xi_{j_h}}{\xi_{i_m}} \right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{i_m}}{I_{j_h}} \right)^{\frac{1}{\eta-1}} \frac{a_T}{d(s_f)H(s_f)} \frac{T-s_f}{T}}, \tag{55b}$$

for $i, j = A, M, S$.

Also, the relative price of two traditional commodities equals

$$p_{i_h} Z_{i_h} \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{i_h}} \right)^{\frac{1}{\eta-1}} \frac{a_T}{d(s_f)H(s_f)} \frac{T}{T-s_f} = p_{j_h} Z_{j_h} \xi_{j_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{j_h}} \right)^{\frac{1}{\eta-1}} \frac{a_T}{d(s_f)H(s_f)} \frac{T}{T-s_f}$$

$$\boxed{\frac{p_{i_h}}{p_{j_h}} = \frac{Z_{j_h}}{Z_{i_h}}\left(\frac{\xi_{j_h}}{\xi_{i_h}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{i_h}}{I_{j_h}}\right)^{\frac{1}{\eta-1}}} \tag{56a}$$

for $i, j = A, M, S$. Note that human capital from s_f does not matter in Equation (56a)

B.3.2 Household commodities and leisure prices

For the price of leisure relative to the traditional consumption good, combining the first order conditions wrt L_l^f & L_l^m, we have that

$$\begin{aligned}
\frac{\frac{\delta L_l}{\delta L_l^f}}{\frac{\delta L_l}{\delta L_l^m}} &= \frac{w_f d(s_f)H(s_f)}{w_m d(s_m)H(s_m)} = x\tilde{d}\tilde{H} \\
\frac{\xi_l \left(L_l^f\right)^{-\frac{1}{\eta_l}} L_l^{\frac{1}{\eta_l}}}{(1-\xi_l)\left(L_l^m\right)^{-\frac{1}{\eta_l}} L_l^{\frac{1}{\eta_l}}} &= \frac{\xi_l}{1-\xi_l}\left(\frac{L_l^m}{L_l^f}\right)^{\frac{1}{\eta_l}} = x\tilde{d}\tilde{H} \\
\frac{L_l^m}{L_l^f} &= \alpha_l^{-\eta_l} x^{\eta_l} (\tilde{d}\tilde{H})^{\eta_l}
\end{aligned} \tag{57}$$

Apply the same trick as before, i.e. define I_l as the female wage bill share in leisure, during working life:

$$I_l := \frac{w_f L_l^f d(s_f)H(s_f)}{w_f L_l^f d(s_f)H(s_f) + w_m L_l^m d(s_m)H(s_m)}. \tag{58a}$$

Then I_l reduces to

$$I_l = \frac{1}{1+\alpha_l^{-\eta_l} x^{\eta_l - 1} (\tilde{d}\tilde{H})^{\eta_l - 1}} \tag{58b}$$

Replace male leisure time by female leisure time in aggregate leisure L_l (see Eq. 46e) yields

$$\begin{aligned}
L_l &= \left[\xi_l (L_l^f)^{\frac{\eta_l - 1}{\eta_l}} + (1-\xi_l)(\alpha_l^{-\eta_l} x^{\eta_l} (\tilde{d}\tilde{H})^{\eta_l} L_l^f)^{\frac{\eta_l - 1}{\eta_l}}\right]^{\frac{\eta_l}{\eta_l - 1}} \\
L_l &= L_l^f \left(\frac{\xi_l}{I_l}\right)^{\frac{\eta_l}{\eta_l - 1}}
\end{aligned} \tag{59}$$

Analogue to the traditional consumption good, require that $\frac{\delta U}{\delta L_l^f} = \lambda p_l$. Rearrange the derivative wrt L_l^f in Equation (47c) for λ and substitute:

$$\lambda = \frac{\delta U}{\delta L_l^f}\frac{1}{p_l} = \frac{a_T}{w_f d(s_f)H(s_f)}\frac{\delta U}{\delta L_l}\frac{\delta L_l}{\delta L_l^f}$$
$$p_l = w_f \frac{d(s_f)H(s_f)}{a_T}\left(\frac{\delta L_l}{\delta L_l^f}\right)^{-1} \tag{60}$$

Compute the derivative of aggregate leisure wrt female leisure time, and replace using Equation (59):

$$\frac{\delta L_l}{\delta L_l^f} = \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}} \tag{61a}$$

Then the leisure price equals

$$p_l = w_f \frac{d(s_f)H(s_f)}{a_T}\left(\xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\right)^{-1} \tag{61b}$$
$$w_f = p_l \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\frac{a_T}{d(s_f)H(s_f)} \tag{61c}$$

Since wages clear across sectors, we can equate Equation (61c) and (54) to get the price of leisure relative to the price of a traditional commodity:

$$p_l \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\frac{a_T}{d(s_f)H(s_f)} = p_{j_h} Z_{j_h} \xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}}\frac{a_T}{d(s_f)H(s_f)}\frac{T-s_f}{T}$$

$$\boxed{\frac{p_l}{p_{j_h}} = Z_{j_h}\xi_{j_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{\eta-1}}\xi_l^{\frac{\eta_l}{1-\eta_l}}\left(\frac{1}{I_l}\right)^{\frac{1}{1-\eta_l}}\frac{T-s_f}{T}} \tag{62a}$$

$$\boxed{\frac{p_{j_h}}{p_l} = Z_{j_h}\xi_{j_h}^{\frac{\eta}{1-\eta}}\left(\frac{1}{I_{j_h}}\right)^{\frac{1}{1-\eta}}\xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\frac{T}{T-s_f}} \tag{62b}$$

And for the price ratio of leisure vs firm commodities, we have

$$p_l \xi_l^{\frac{\eta_l}{\eta_l-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{\eta_l-1}}\frac{a_T}{d(s_f)H(s_f)} = p_{j_m} Z_{j_m} \xi_{j_m}^{\frac{\eta}{\eta-1}} I_{j_m}^{\frac{1}{1-\eta}}$$

$$\boxed{\frac{p_l}{p_{j_m}} = Z_{j_m} \xi_{j_m}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \left(\frac{1}{\xi_l}\right)^{\frac{\eta_l}{\eta_l-1}} I_l^{\frac{1}{\eta_l-1}} \frac{d(s_f)H(s_f)}{a_T}} \tag{62c}$$

B.3.3 Relative Expenditures

Expenditure on firm versus household commodities

Derive $MRS_{c_{jm},c_{j_h}}$: We combine derivatives wrt c_{j_m} and $L^f_{j_h}$ (see Equations (47a) and (47b)), to have marginal rates of substitution between formal vs traditional services:

$$MRS_{c_{jm},c_{j_h}} = \frac{\frac{\delta U}{\delta c_{jm}}}{\frac{\delta U}{\delta c_{j_h}}} = \frac{\lambda a_T p_{j_m}}{\lambda d(s_f)H(s_f)w_f} \frac{\delta c_{j_h}}{\delta L^f_{j_h}} = \frac{p_{j_m}}{w_f} \frac{a_T}{d(s_f)H(s_f)} \frac{\delta c_{j_h}}{\delta L^f_{j_h}}$$

Note that the derivative expansions needed from the chain rule on the LHS cancel out, such that

$$\frac{\frac{\delta U}{\delta c}\frac{\delta c}{\delta c_j}\frac{\delta c_j}{\delta c_{jm}}}{\frac{\delta U}{\delta c}\frac{\delta c}{\delta c_j}\frac{\delta c_j}{\delta c_{j_h}}} = \frac{\frac{\delta c_j}{\delta c_{jm}}}{\frac{\delta c_j}{\delta c_{j_h}}} = \frac{\psi_j c_{j_m}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{(1-\psi_j) c_{j_h}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}$$

$$\frac{\psi_j}{1-\psi_j}\left(\frac{c_{jm}}{c_{j_h}}\right)^{-\frac{1}{\sigma}} = \frac{p_{j_m}}{w_f} \frac{a_T}{d(s)H(s_f)} \frac{\delta c_{j_h}}{\delta L^f_{j_h}}$$

We know w_f from Equation (52a) that related the implicit price of traditional services to wages. Substitute and get

$$\frac{\psi_j}{1-\psi_j}\left(\frac{c_{jm}}{c_{j_h}}\right)^{-\frac{1}{\sigma}} = \frac{p_{j_m}}{p_{j_h}} \frac{\frac{a_T}{d(s_f)H(s_f)} \frac{\delta c_{j_h}}{\delta L^f_{j_h}}}{\frac{a_T}{d(s_f)H(s_f)} \frac{\delta c_{j_h}}{\delta L^f_{j_h}}} = \frac{p_{j_m}}{p_{j_h}}$$

$$\left(\frac{c_{jm}}{c_{j_h}}\right)^{\frac{1}{\sigma}} = \frac{p_{j_h}}{p_{j_m}} \frac{\psi_j}{1-\psi_j}$$

$$\boxed{\frac{c_{jm}}{c_{j_h}} = \left(\frac{p_{j_h}}{p_{j_m}}\right)^{\sigma} \left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma}} \tag{63}$$

Relative expenditures on formal relative to traditional commodities then are:

$$\boxed{E_{jmh} = \frac{p_{j_m} c_{j_m}}{p_{j_h} c_{j_h}} = \left(\frac{p_{j_h}}{p_{j_m}}\right)^{\sigma-1} \left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma_j}} \tag{64a}$$

Making use of our results for relative prices p_{j_h}/p_{j_m} in Equation (55a)

$$\frac{p_{j_h}}{p_{j_m}} = \frac{Z_{j_m}}{Z_{j_h}} \left(\frac{\xi_{j_m}}{\xi_{j_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \frac{d(s_f)H(s_f)}{a_T} \frac{T}{T-s_f} \tag{64b}$$

and defining $\hat{Z}_{jmh} := \frac{Z_{j_m}}{Z_{j_h}} \left(\frac{\psi_j}{1-\psi_j}\right)^{\frac{\sigma}{\sigma-1}}$

$$E_{jmh} = \left[\frac{Z_{j_m}}{Z_{j_h}} \left(\frac{\xi_{j_m}}{\xi_{j_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}} \frac{d(s_f)H(s_f)}{a_T} \frac{T}{T-s_f}\right]^{\sigma-1} \left(\frac{\psi_j}{1-\psi_j}\right)^{\sigma}$$

Eventually, the relative expenditure share of formal to traditional commodity j equals

$$E_{jmh} = \hat{Z}_{jmh}^{\sigma-1} \left[\left(\frac{\xi_{j_m}}{\xi_{j_h}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_h}}{I_{j_m}}\right)^{\frac{1}{\eta-1}} H(s_f) \frac{d(s_f)}{a_T} \frac{T}{T-s_f}\right]^{\sigma-1}. \tag{64c}$$

Expenditures across commodity kinds

For firm commodities j_m, i_m, derive $MRS_{c_{j_m},c_{i_m}}$. Combine the derivatives wrt c_{j_m} and c_{i_m} (see Equation 47a):

$$MRS_{c_{j_m},c_{i_m}} = \frac{p_{j_m}}{p_{i_m}} = \frac{\frac{\delta U}{\delta c}\frac{\delta c}{\delta c_j}\frac{\delta c_j}{\delta c_{j_m}}}{\frac{\delta U}{\delta c}\frac{\delta c}{\delta c_i}\frac{\delta c_i}{\delta c_{i_m}}} = \frac{\omega_j}{\omega_i} \frac{c_j^{-\frac{1}{\varepsilon}} \psi_j c_{j_m}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{c_i^{-\frac{1}{\varepsilon}} \psi_i c_{i_m}^{-\frac{1}{\sigma}} c_i^{\frac{1}{\sigma}}}$$

$$\frac{p_{j_m}}{p_{i_m}} = \frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\left(\frac{c_i}{c_j}\right)^{\frac{1}{\varepsilon}}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{1}{\sigma}}\left(\frac{c_j}{c_{j_m}}\right)^{\frac{1}{\sigma}}$$

$$\left(\frac{p_{j_m}}{p_{i_m}}\right)^{\varepsilon} = \frac{c_{i_m}}{c_{j_m}}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\varepsilon-\sigma}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\sigma-\varepsilon}{\sigma}}\left(\frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\right)^{\varepsilon}$$

$$\frac{c_{j_m}}{c_{i_m}} = \left(\frac{p_{i_m}}{p_{j_m}}\right)^{\varepsilon}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\varepsilon-\sigma}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\sigma-\varepsilon}{\sigma}}\left(\frac{\omega_j}{\omega_i}\frac{\psi_j}{\psi_i}\right)^{\varepsilon} \tag{65a}$$

and

$$\frac{c_{i_m}}{c_{j_m}} = \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\varepsilon}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\sigma-\varepsilon}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\varepsilon-\sigma}{\sigma}}\left(\frac{\omega_i}{\omega_j}\frac{\psi_i}{\psi_j}\right)^{\varepsilon} \tag{65b}$$

Problematic: The c_i, c_j in Equation (65b). Consider the inverse relative demand c_{i_m}/c_{i_h}

$$\frac{c_{i_m}}{c_{i_h}} = \left(\frac{p_{i_h}}{p_{i_m}}\right)^\sigma \left(\frac{\psi_i}{1-\psi_i}\right)^\sigma$$

in Equation (63) to derive c_i as a function of c_{i_m}:

$$c_i = \left[\psi_i c_{i_h}^{\frac{\sigma-1}{\sigma}} + (1-\psi_i) c_{i_m}^{\frac{\sigma-1}{\sigma}} \left(\frac{p_{i_m}}{p_{i_h}} \frac{1-\psi_i}{\psi_i}\right)^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}}$$

$$\frac{c_i}{c_{i_m}} = \psi_i^{\frac{\sigma}{\sigma-1}} \left[1 + \underbrace{\left(\frac{1-\psi_i}{\psi_i}\right)^\sigma \left(\frac{p_{i_m}}{p_{i_h}}\right)^{\sigma-1}}_{=E_{i_{mh}}^{-1}}\right]^{\frac{\sigma}{\sigma-1}} \tag{66a}$$

Note that the inner part of Equation (66a) equals the inverse of $E_{i_{mh}}$ in Equation (64a), so that one can write

$$\boxed{\frac{c_i}{c_{i_m}} = \psi_i^{\frac{\sigma}{\sigma-1}} \left(1 + \frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma}{\sigma-1}}} \tag{66b}$$

This equation holds for $i = A, M, S$, i.e. we also get

$$\boxed{\frac{c_j}{c_{j_m}} = \psi_j^{\frac{\sigma}{\sigma-1}} \left(1 + \frac{1}{E_{j_{mh}}}\right)^{\frac{\sigma}{\sigma-1}}} \tag{67}$$

Plug into Equation (65a), which yields

$$\frac{c_{j_m}}{c_{i_h}} = \left(\frac{p_{i_h}}{p_{j_m}}\right)^\varepsilon \left(1 + \frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}} \left(1 + \frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}} \left(\frac{\omega_i}{\omega_j}\right)^\varepsilon \psi_i^{\sigma\frac{\varepsilon-1}{1-\sigma}} \psi_j^{\sigma\frac{1-\varepsilon}{1-\sigma}} \tag{68}$$

and since it will be useful to have everything in terms of relative expenditures, again write

$$\frac{p_{j_m} c_{j_m}}{p_{i_m} c_{i_m}} = E_{j_m i_m} = \left(\frac{p_{i_m}}{p_{j_m}}\right)^{\varepsilon-1} \left(1 + \frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}} \left(1 + \frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}} \left(\frac{\omega_i}{\omega_j}\right)^\varepsilon \psi_i^{\sigma\frac{\varepsilon-1}{1-\sigma}} \psi_j^{\sigma\frac{1-\varepsilon}{1-\sigma}} \tag{69a}$$

Again, make use of the relative price equation from the formal side (see Equation (45k)) to substitute for p_{i_m}/p_{j_m} and define

$$\hat{Z}_{j_m i_m} := \frac{Z_{j_m}}{Z_{i_m}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{\varepsilon}{\varepsilon-1}} \psi_i^{\frac{\sigma}{1-\sigma}} \psi_j^{\frac{\sigma}{\sigma-1}} \tag{69b}$$

to get

$$E_{j_m i_m} = \hat{Z}_{j_m i_m}^{\varepsilon-1} \left[\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_m}}\right)^{\frac{1}{1-\eta}} \right]^{\varepsilon-1} \left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}} \left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}} \tag{69c}$$

Leisure:

$MRS_{L_l, c_{j_m}}$. Equate marginal utilities, set equal to price ratio. Note that

$$\frac{\delta U}{\delta L_l} = \lambda w_f \frac{d(s_f)H(s_f)}{a_T} \left(\frac{\delta L_l}{\delta L_{fl}}\right)^{-1} = \lambda p_l$$

by our definition of the implicit leisure price (see Equation (60)). Thus

$$\frac{\frac{\delta U}{\delta c_{j_m}}}{\frac{\delta U}{\delta L_l}} = \frac{p_{j_m}}{p_l}$$

$$\frac{\frac{1}{c-\bar{c}} \omega_j c_j^{-\frac{1}{\varepsilon}} c^{\frac{1}{\varepsilon}} \psi_j c_{j_m}^{-\frac{1}{\sigma}} c_j^{\frac{1}{\sigma}}}{\frac{\varphi}{L_l}} = \frac{p_{j_m}}{p_l}$$

$$\frac{1}{\varphi} \frac{c}{c-\bar{c}} \omega_j \psi_j \left(\frac{c}{c_j}\right)^{\frac{1-\varepsilon}{\varepsilon}} \left(\frac{c_j}{c_{j_m}}\right)^{\frac{1-\sigma}{\sigma}} = \frac{p_{j_m} c_{j_m}}{p_l L_l}$$

We know the relations c_j / c_{j_m} and c_i / c_{i_m} from Equation (67) and Equation (66b)

$$\frac{c_j}{c_{j_m}} = \psi_j^{\frac{\sigma}{\sigma-1}} \left(1+\frac{1}{E_{j_m j_h}}\right)^{\frac{\sigma}{\sigma-1}}$$

Also,

$$c = \left[\sum_{i=A,M,S} \omega_i (c_i)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{70a}$$

$$c = \left[\omega_j (c_j)^{\frac{\varepsilon-1}{\varepsilon}} \sum_{i=A,M,S} \frac{\omega_i}{\omega_j} \left(\frac{c_i}{c_j}\right)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}, j = A, M, S \tag{70b}$$

$$c = \left[\omega_j (c_j)^{\frac{\varepsilon-1}{\varepsilon}} \sum_{i=A,M,S} \frac{\omega_i}{\omega_j} \left(\frac{c_i}{c_{i_m}} \frac{c_{i_m}}{c_{j_m}} \frac{c_{j_m}}{c_j}\right)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{70c}$$

$$\frac{c}{c_j} = \left[\omega_j \sum_{i=A,M,S} \frac{\omega_i}{\omega_j} \left(\frac{c_i}{c_{i_m}} \frac{c_{i_m}}{c_{j_m}} \frac{c_{j_m}}{c_j}\right)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{70d}$$

Let's simplify the expression containing multiple relative demands, using Equation (65b):

$$\frac{c_{i_m}}{c_{j_m}}\frac{c_i}{c_{i_m}}\frac{c_{j_m}}{c_j} = \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\varepsilon}\left(\frac{c_{i_m}}{c_i}\right)^{\frac{\sigma-\varepsilon}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\varepsilon-\sigma}{\sigma}}\left(\frac{\omega_i}{\omega_j}\frac{\psi_i}{\psi_j}\right)^{\varepsilon}\frac{c_i}{c_{i_m}}\frac{c_{j_m}}{c_j}$$

$$= \left(\frac{p_{j_m}}{p_{i_m}}\right)^{\varepsilon}\left(\frac{c_i}{c_{i_m}}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{c_{j_m}}{c_j}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{\omega_i}{\omega_j}\frac{\psi_i}{\psi_j}\right)^{\varepsilon}$$

$$\frac{c_i}{c_j} = \left[\left(\frac{Z_{j_m}}{Z_{i_m}}\psi_i^{\frac{\sigma}{1-\sigma}}\psi_j^{\frac{\sigma}{\sigma-1}}\right)^{\varepsilon}\right]^{-1}\left(\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{i_m}}{I_{j_m}}\right)^{\frac{1}{\eta-1}}\right)^{-\varepsilon}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\varepsilon}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon}{1-\sigma}}\left(\frac{\omega_i}{\omega_j}\right)^{\varepsilon} \tag{71a}$$

Remember that we had defined in Equation (69c)

$$\hat{Z}_{j_m i_m} := \frac{Z_{j_m}}{Z_{i_m}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}}\psi_i^{\frac{\sigma}{1-\sigma}}\psi_j^{\frac{\sigma}{\sigma-1}}$$

$$E_{j_m i_m} = \hat{Z}_{j_m i_m}^{\varepsilon-1}\left[\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{j_m}}{I_{i_m}}\right)^{\frac{1}{1-\eta}}\right]^{\varepsilon-1}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\sigma-\varepsilon}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon-\sigma}{\sigma-1}}$$

Thus, Equation (71a) simplifies to

$$\frac{c_i}{c_j} = \left[\left(\hat{A}_{j_m i_m}\left(\frac{\xi_{j_m}}{\xi_{i_m}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{I_{i_m}}{I_{j_m}}\right)^{\frac{1}{\eta-1}}\right)^{\varepsilon}\right]^{-1}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\varepsilon}{\sigma-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon}{1-\sigma}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\frac{c_i}{c_j} = \left(E_{j_m i_m}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(1+\frac{1}{E_{i_{mh}}}\right)^{\frac{\varepsilon}{\varepsilon-1}}\left(1+\frac{1}{E_{j_{mh}}}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\boxed{\frac{c_i}{c_j} = \left(E_{j_m i_m}\frac{E_{i_{mh}}}{E_{j_{mh}}}\frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}}} \tag{72}$$

We then have that

$$\frac{c}{c_j} = \left[\omega_j\sum_{i=A,M,S}\frac{\omega_i}{\omega_j}\left(\frac{c_i}{c_j}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\frac{c}{c_j} = \omega_j^{\frac{\varepsilon}{\varepsilon-1}}\left[\sum_{i=A,M,S}\frac{\omega_i}{\omega_j}\left(\left(E_{j_m i_m}\frac{E_{i_{mh}}}{E_{j_{mh}}}\frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(\frac{\omega_j}{\omega_i}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\frac{c}{c_j} = \omega_j^{\frac{\varepsilon}{\varepsilon-1}}\left[1+\sum_{i\neq j}\left(E_{j_m i_m}\frac{E_{i_{mh}}}{E_{j_{mh}}}\frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{-1}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{73a}$$

The expression then becomes

$$\frac{p_{j_m} c_{j_m}}{p_l L_l} = \frac{1}{\varphi} \frac{c}{c - \bar{c}} \omega_j \psi_j \left(\omega_j^{\frac{\varepsilon}{\varepsilon-1}} \left[1 + \sum_{i \neq j} \left(E_{j_m i_m} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1 + E_{j_{mh}}}{1 + E_{i_{mh}}} \right)^{-1} \right]^{\frac{\varepsilon}{\varepsilon-1}} \right)^{\frac{1-\varepsilon}{\varepsilon}} \left(\frac{c_j}{c_{j_m}} \right)^{\frac{1-\sigma}{\sigma}}$$

$$\frac{p_{j_m} c_{j_m}}{p_l L_l} = \frac{1}{\varphi} \frac{c}{c - \bar{c}} \psi_j \left[1 + \sum_{i \neq j} \frac{1}{E_{j_m i_m}} \frac{E_{j_{mh}}}{E_{i_{mh}}} \frac{1 + E_{i_{mh}}}{1 + E_{j_{mh}}} \right]^{-1} \left(\psi_j^{\frac{\sigma}{\sigma-1}} \left(1 + \frac{1}{E_{j_{mh}}} \right)^{\frac{\sigma}{\sigma-1}} \right)^{\frac{1-\sigma}{\sigma}}$$

$$\boxed{E_{j_m l} = \frac{1}{\varphi} \frac{c}{c - \bar{c}} \left[1 + \sum_{i \neq j} \frac{1}{E_{j_m i_m}} \frac{E_{j_{mh}}}{E_{i_{mh}}} \frac{1 + E_{i_{mh}}}{1 + E_{j_{mh}}} \right]^{-1} \left(1 + \frac{1}{E_{j_{mh}}} \right)^{-1}} \tag{74}$$

$$\boxed{E_{l j_m} = \varphi \frac{c - \bar{c}}{c} \left(1 + \sum_{i \neq j} \frac{1}{E_{j_m i_m}} \frac{E_{j_{mh}}}{E_{i_{mh}}} \frac{1 + E_{i_{mh}}}{1 + E_{j_{mh}}} \right) \left(1 + \frac{1}{E_{j_{mh}}} \right)} \tag{75}$$

B.4 A root function for the wage gap

B.4.1 Rewriting the Budget Constraint

Putting it all together: Rewrite the BC using the implicit price for home consumption good:

$$\begin{aligned}
\int_{t=0}^{T} e^{-\rho t}\Big(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j\Big)dt &= \sum_{g=m,f}\int_{t=s_g}^{T} e^{-\rho t}\Big[w_g H(s_g)(L^g - \sum_{j=A,M,S,l} L^g_{j_h})\Big]dt \\
a_T\Big(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j\Big) &= \sum_{g=m,f} d(s_g)\Big[w_g H(s_g)(L^g - \sum_{j=A,M,S,l} L^g_{j_h})\Big]
\end{aligned}$$

Take time in non-market production and leisure to the LHS:

$$a_T\Big(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j + \sum_{g=m,f} w_g \frac{d(s_g)H(s_g)}{a_T}\sum_{i=A_h,M_h,S_h,l} L^g_i\Big) = \sum_{g=m,f} w_g d(s_g)H(s_g)L^g$$

replace male by female leisure/time in home production using Equation (48d):

$$L^m_{j_h} = \alpha_{j_h}^{-\eta} x^{\eta}(\tilde{d}\tilde{H})^{\eta}\tilde{T}^{1-\eta}L^f_{j_h}, \tag{76a}$$

so that the LHS takes the form:

$$\begin{aligned}
&a_T\Big(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j + w_m\frac{d(s_m)H(s_m)}{a_T}\sum_{\substack{i=A_h,M_h\\ S_h,l}} L^m_i + w_f\frac{d(s_f)H(s_f)}{a_T}\sum_{\substack{i=A_h,M_h\\ S_h,l}} L^f_i\Big) \\
&a_T\Big(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j + w_f\frac{d(s_f)H(s_f)}{a_T}\sum_{\substack{i=A_h,M_h\\ S_h,l}} L^f_i \underbrace{\Big[1+\alpha_i^{-\eta}(x\tilde{d}\tilde{H})^{\eta-1}\tilde{T}^{1-\eta}\Big]}_{\text{see Equations 49c, 58b}}\Big).
\end{aligned} \tag{76b}$$

I get

$$a_T\Big(\sum_{\substack{j=A_m,\\ M_m,S_m}} p_j c_j + w_f\frac{d(s_f)H(s_f)}{a_T}\sum_{\substack{i=A_h,M_h\\ S_h,l}} L^f_i I_i^{-1}\Big) = \sum_{g=m,f} w_g d(s_g)H(s_g)L^g \tag{76c}$$

Now remember the definition of implicit prices, given by Equations (52a) & (60):

$$p_{j_h} = w_f\frac{d(s_f)H(s_f)}{a_T}\Big(\frac{\delta c_{j_h}}{\delta L^f_{j_h}}\Big)^{-1}, j = A, M, S$$

which is substituted into the BC:

$$a_T\left(\sum_{\substack{j=A_m,\\M_m,S_m}} p_j c_j + \sum_{\substack{i=A_h,M_h\\S_h,l}} p_i L_i^f I_i^{-1} \frac{\delta c_i}{\delta L_i^f}\right) = \sum_{g=m,f} w_g d(s_g) H(s_g) L^g \tag{76d}$$

Make use of the partial derivatives (see Equations (53a) (and similar for leisure l, 61a)

$$\frac{\delta c_{i_h}}{\delta L_{i_h}^f} = Z_{i_h} \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \frac{T-s_f}{T} \tag{76e}$$

and Equations (50), (59) :

$$\frac{L_{j_h}}{L_{j_h}^f} = \left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}} (T-s_f), j = A, M, S \tag{76f}$$

yields

$$a_T\left(\sum_{\substack{j=A_m,\\M_m,S_m}} p_j c_j + \sum_{\substack{i=A_h,M_h\\S_h,l}} p_i L_i^f I_i^{-1} Z_i \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_i}\right)^{\frac{1}{\eta-1}} \frac{T-s_f}{T}\right) = \sum_{g=m,f} w_g d(s_g) H(s_g) L^g$$

$$a_T\left(\sum_{\substack{j=A_m,\\M_m,S_m}} p_j c_j + \sum_{\substack{i=A_h,M_h\\S_h,l}} p_i \frac{Z_i}{T} L_i\right) = \sum_{g=m,f} w_g d(s_g) H(s_g) L^g \tag{76g}$$

s.t. we have a rewritten lifetime budget constraint of the form

$$a_T\left(\sum_{\substack{j=A_m,\\M_m,S_m}} p_j c_j + \sum_{\substack{i=A_h,M_h\\S_h,l}} p_i c_i + p_l L_l\right) = \sum_{g=m,f} w_g d(s_g) H(s_g) L^g \tag{76h}$$

The idea now is to derive a condition including expenditure shares relative to non-market services $j_h = S_h$

$$a_T\left(\sum_{\substack{j=A_m,\\M_m,S_m}} \frac{p_j c_j}{p_{S_h} c_{S_h}} + \sum_{\substack{i=A_h,M_h\\S_h,l}} \frac{p_i c_i}{p_{S_h} c_{S_h}} + \frac{p_l L_l}{p_{S_h} c_{S_h}}\right) = \sum_{g=m,f} \frac{w_g d(s_g) H(s_g) L^g}{p_{S_h} c_{S_h}}$$

$$a_T\left(\sum_{j=A,M,S} \sum_{k=m,n} E_{i_k S_h} + E_{l S_h}\right) = \sum_{g=m,f} \frac{w_g d(s_g) H(s_g) L^g}{p_{S_h} c_{S_h}}$$

$$a_T\left(\sum_{j=A,M,S}\sum_{k=m,n} E_{i_kS_h}+E_{lS_h}\right)\frac{1}{w_f d(s_f)H(s_f)L^f}=\sum_{g=m,f}\frac{w_g d(s_g)H(s_g)L^g}{p_{S_h}c_{S_h}w_f d(s_f)H(s_f)L^f}$$

$$\left(a_T\sum_{j=A,M,S}\sum_{k=m,n} E_{i_kS_h}+E_{lS_h}\right)^{-1} w_f d(s_f)H(s_f)L^f = p_{S_h}c_{S_h}\frac{w_f d(s_f)H(s_f)L^f}{\sum_{g=m,f} w_g d(s_g)H(s_g)L^g} \tag{77a}$$

Now note that the RHS of Equation (77a) contains the expression for the ratio of female in total HH lifetime earnings. We can rewrite this part as

$$\begin{aligned}
I_L &= \frac{w_f L^f d(s_f)H(s_f)}{L^f w_f H(s_f)d(s_f)+w_m L^m d(s_m)H(s_m)}\\
&= \frac{w_f L^f d(s_f)H(s_f)}{w_f L^f d(s_f)H(s_f)\left(1+\frac{L^m w_m H(s_m)d(s_m)}{L^f w_f H(s_f)d(s_f)}\right)}\\
I_L &= \frac{1}{1+\left(\tilde{L}x\tilde{d}\tilde{H}\right)^{-1}}
\end{aligned} \tag{78}$$

Use the implicit definition of p_{S_h}, derived in Equation (52a):

$$p_{j_h} = w_f\frac{d(s_f)H(s_f)}{a_T}\left(\frac{\delta c_{j_h}}{\delta L^f_{j_h}}\right)^{-1},$$

to replace p_{S_h} in Equation (77a)

$$\left(a_T\sum_{j=A,M,S}\sum_{k=m,n} E_{i_kS_h}+E_{lS_h}\right)^{-1} w_f d(s_f)H(s_f)L^f = p_{S_h}c_{S_h}I_L$$

$$\left(a_T\sum_{j=A,M,S}\sum_{k=m,n} E_{i_kS_h}+E_{lS_h}\right)^{-1} w_f d(s_f)H(s_f)L^f = w_f\frac{d(s_f)H(s_f)}{a_T}\left(\frac{\delta c_{S_h}}{\delta L^f_{S_h}}\right)^{-1}c_{S_h}I_L$$

$$\left(\sum_{j=A,M,S}\sum_{k=m,n} E_{i_kS_h}+E_{lS_h}\right)^{-1} L^f = \left(\frac{\delta c_{S_h}}{\delta L^f_{S_h}}\right)^{-1}c_{S_h}I_L \tag{79a}$$

Now use Equations (53a) as well as (50) to replace L_{S_h} in $c_{S_h}=\frac{1}{T}Z_{S_h}L_{S_h}$

$$\frac{1}{\sum_{j=A,M,S}\sum_{k=m,n} E_{i_kS_h}+E_{lS_h}}L^f = \left[Z_{S_h}\xi_{i_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{S_h}}\right)^{\frac{1}{\eta-1}}\frac{T-s_f}{T}\right]^{-1}c_{S_h}I_L$$

$$\frac{1}{\sum_{j=A,M,S}\sum_{k=m,n} E_{i_kS_h}+E_{lS_h}}L^f = \frac{1}{I_{S_h}}L^f_{S_h}I_L \tag{80a}$$

Or

$$\frac{L^f_{S_h}}{L^f} = \frac{1}{I_L \sum\limits_{\substack{j=A_m,M_m,S_m \\ A_h,M_h,S_h,l}} \frac{E_{jS_h}}{I_{S_h}}} \tag{80b}$$

B.4.2 Rewriting the Time Constraint

Use market clearing conditions: Demand equals Supply on market & non-market production sectors

- Market sectors $j = A_m, M_m, S_m$

$$Tc_j = Z_j L_j, \tag{81a}$$

- Non-market sectors, $j = A_h, M_h, S_h$

$$Tc_j = Z_j L_j, \tag{81b}$$

Time in home and market production cannot exceed total time endowment:

$$L^g_{A_m} + L^g_{M_m} + L^g_{S_m} = L^g - L^g_{A_h} + L^g_{M_h} + L^g_{S_h} - L^g_l \tag{81c}$$

I now derive a similar condition to that in Equation (80b) by using market clearing conditions in expressions for (per period) relative expenditures of two commodities $j,k = A_m, A_h, M_m, M_h, S_m, S_h, l$, which we defined as

$$E_{kj} = \frac{p_k c_k}{p_j c_j}$$

Formal to formal.

Using Equation (81a), we can derive for market sectors $j,k = A_m, M_m, S_m$ that

$$E_{kj} = \frac{p_k c_k}{p_j c_j} = \frac{p_k Z_k}{p_j Z_j} \frac{L_k \frac{1}{T}}{L_j \frac{1}{T}}.$$

From the relative price Equation (45k), we remember that we can substitute for $p_k Z_k / p_j Z_j$:

$$\frac{p_k Z_k}{p_j Z_j} \frac{L_k}{L_j} = \left(\frac{\xi_j}{\xi_k}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_k(x)}{I_j(x)}\right)^{\frac{1}{\eta-1}} \frac{L_k}{\tilde{L}_j} \tag{82a}$$

Extend the fraction $\frac{L_k}{L_j} = \frac{\frac{L_k}{L_k^f} L_k^f}{\frac{L_j}{L_j^f} L_j^f}$ and replace L_j, L_k using Equation (45i):

$$\frac{L_j}{L_j^f} = \left(\frac{\xi_j}{I_j}\right)^{\frac{\eta}{\eta-1}} (T - s_f) H(s_f); \tag{82b}$$

yields

$$
\begin{aligned}
E_{kj}(x) &= \left(\frac{\xi_j}{\xi_k}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_k(x)}{I_j(x)}\right)^{\frac{1}{\eta-1}} \frac{\left(\frac{\xi_k}{I_k}\right)^{\frac{\eta}{\eta-1}}(T-s_f)H(s_f)}{\left(\frac{\xi_j}{I_j}\right)^{\frac{\eta}{\eta-1}}(T-s_f)H(s_f)} \frac{L_k^f}{L_j^f} \\
&= \left(\frac{\xi_j}{\xi_k}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_k}{I_j}\right)^{\frac{1}{\eta-1}} \left(\frac{\xi_k}{\xi_j}\frac{I_j(x)}{I_k(x)}\right)^{\frac{\eta}{\eta-1}} \frac{L_k^f}{L_j^f} \\
&= \left(\frac{I_j}{I_k}\right)^{\frac{\eta-1}{\eta-1}} \frac{L_k^f}{L_j^f}
\end{aligned}
$$

such that for two market commodities, we have

$$
\frac{L_k^f}{L_j^f} = E_{kj}\frac{I_k}{I_j} \tag{82c}
$$

Formal to traditional.

Expenditures on commodities produced in the market sector $j_m = \{A_m, M_m, S_m\}$ relative to non-market commodities $i_h = \{A_h, M_h, S_h\}$ equal

$$
\begin{aligned}
E_{j_m i_h} &= \frac{p_{j_m} c_{j_m}}{p_{i_h} c_{i_h}} \\
&= \frac{p_{j_m} Z_{j_m} \frac{1}{T}}{p_{i_h} Z_{i_h} \frac{1}{T}} \frac{L_{j_m}}{L_{j_h}}
\end{aligned}
$$

Similar to our above derivation of relative market sector expenditures, we make use of the relative price Equation (55b) to substitute

$$
\frac{p_{j_m}}{p_{i_h}} = \frac{Z_{i_h}}{Z_{j_m}} \left(\frac{\xi_{i_h}}{\xi_{j_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \frac{a_T}{d(s_f)H(s_f)} \frac{T-s_f}{T} \tag{83a}
$$

yields

$$
E_{j_m i_h} = \left(\frac{\xi_{i_h}}{\xi_{j_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \frac{a_T}{d(s_f)H(s_f)} \frac{T-s_f}{T} \frac{L_{j_m}}{L_{j_h}} \tag{83b}
$$

We substitute for L_{j_m} and L_{i_h} using Equations (45i) $L_{j_m} = \left(\frac{\xi_{j_m}}{I_{j_m}}\right)^{\frac{\eta}{\eta-1}}(T-s_f)H(s_f)L_{j_m}^f$

and 50: $L_{i_h} = \left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} (T - s_f) L^f_{i_h}$:

$$E_{j_m i_h} = \left(\frac{\xi_{i_h}}{\xi_{j_m}}\right)^{\frac{\eta}{\eta-1}} \left(\frac{I_{j_m}}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \frac{a_T}{d(s_f)H(s_f)} \frac{T - s_f}{T} \frac{\left(\frac{\xi_{j_m}}{I_{j_m}}\right)^{\frac{\eta}{\eta-1}} (T - s_f) H(s_f) L^f_{j_m}}{\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} (T - s_f) L^f_{i_h}}$$

$$E_{j_m i_h} = \frac{I_{i_h}}{I_{j_m}} \frac{a_T}{d(s_f)} \frac{T - s_f}{T} \frac{L^f_{j_m}}{L^f_{i_h}} \tag{83c}$$

s.t. we have relative labor supply equaling

$$\frac{L^f_{j_m}}{L^f_{i_h}} = E_{j_m i_h} \frac{I_{j_m}}{I_{i_h}} \frac{d(s_f)}{a_T} \frac{T}{T - s_f} \tag{83d}$$

Leisure to traditional.
Keep in mind that

$$L_l = L^f_l \left(\frac{\xi_l}{I_l}\right)^{\frac{\eta_l}{\eta_l-1}}$$

$$L_{i_h} = \left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} (T - s_f) L^f_{i_h}$$

Expenditures on leisure relative to the home produced good equal

$$E_{l i_h} = \frac{p_l L_l}{p_{i_h} c_{i_h}}$$

$$E_{l i_h} = \frac{T}{T - s_f} \frac{p_l}{p_{i_h} Z_{i_h}} \frac{L^f_l}{L^f_{i_h}} \frac{\left(\frac{\xi_l}{I_l}\right)^{\frac{\eta_l}{\eta_l-1}}}{\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}}}$$

Substitute for the terms $\frac{p_l}{p_{i_h} Z_{i_h}}$ using the relative price equation Equation (62a):

$$\frac{p_l}{p_{i_h} Z_{i_h}} = \xi_{i_h}^{\frac{\eta}{\eta-1}} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \xi_l^{\frac{\eta_l}{1-\eta_l}} \left(\frac{1}{I_l}\right)^{\frac{1}{1-\eta_l}} \frac{T - s_f}{T} \tag{84a}$$

yields

$$
\begin{aligned}
E_{li_h} &= \frac{T}{T-s_f}\xi_{i_h}^{\frac{\eta}{\eta-1}}\left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}}\xi_l^{\frac{\eta_l}{1-\eta_l}}\left(\frac{1}{I_l}\right)^{\frac{1}{1-\eta_l}}\frac{T-s_f}{T}\frac{L_l^f}{L_{i_h}^f}\frac{\left(\frac{\xi_l}{I_l}\right)^{\frac{\eta_l}{\eta_l-1}}}{\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}}} \\
&= \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}}\left(\frac{1}{I_l}\right)^{\frac{1}{1-\eta_l}}\frac{L_l^f}{L_{i_h}^f}\frac{\left(\frac{1}{I_l}\right)^{\frac{\eta_l}{\eta_l-1}}}{\left(\frac{1}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}}} \\
&= \frac{I_{i_h}}{I_l}\frac{L_l^f}{L_{i_h}^f}
\end{aligned}
$$

We thus have the relation between home production hours and leisure:

$$
\frac{L_l^f}{L_{i_h}^f} = E_{li_h}\frac{I_l}{I_{i_h}} \tag{84b}
$$

Traditional to traditional.

For the relation of hours in the production of one non-market commodity to another non-market commodity, define

$$
\begin{aligned}
E_{j_h i_h} &= \frac{p_{j_h}c_{j_h}}{p_{i_h}c_{i_h}} \\
&= \frac{p_{j_h}\left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}}(T-s_f)L_{j_h}^f}{p_{i_h}\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}}(T-s_f)L_{i_h}^f} \\
&= \frac{I_{i_h}}{I_{j_h}}\frac{L_{j_h}^f}{L_{i_h}^f}
\end{aligned} \tag{85a}
$$

s.t. we also have

$$
\frac{L_{j_h}^f}{L_{i_h}^f} = E_{j_h i_h}\frac{I_{j_h}}{I_{i_h}} \tag{85b}
$$

Finally, we can use the expressions in (Equation (82c)), Equation (83d), Equation (84b) and Equation (85b) to substitute for female labor supplies in the market goods and services sector in the total time endowment Equation (81c):

$$
\begin{aligned}
L^f &= L^f_{A_m} + L^f_{M_m} + L^f_{S_m} + L^f_{A_h} + L^f_{M_h} + L^f_{S_h} + L^f_l \\
\frac{L^f}{L^f_{i_h}} &= \frac{L^f_{A_m}}{L^f_{i_h}} + \frac{L^f_{M_m}}{L^f_{i_h}} + \frac{L^f_{S_m}}{L^f_{i_h}} + \frac{L^f_{A_h}}{L^f_{i_h}} + \frac{L^f_{M_h}}{L^f_{i_h}} + \frac{L^f_{S_h}}{L^f_{i_h}} + \frac{L^f_l}{L^f_{i_h}} \\
\frac{L^f}{L^f_{i_h}} &= \sum_{j=A,M,S} \left[\frac{L^f_{j_m}}{L^f_{i_h}}\right] + \sum_{j=A,M,S} \left[\frac{L^f_{j_h}}{L^f_{i_h}}\right] + \frac{L^f_l}{L^f_{i_h}} \\
\frac{L^f}{L^f_{i_h}} &= \frac{d(s_f)}{a_T} \frac{T}{T-s_f} \sum_{j=A,M,S} \left[E_{j_m i_h} \frac{I_{j_m}}{I_{i_h}}\right] + \sum_{j=A,M,S} \left[E_{j_h i_h} \frac{I_{j_h}}{I_{i_h}}\right] + E_{l i_h} \frac{I_l}{I_{i_h}}
\end{aligned}
\tag{85c}
$$

Such that eventually I get

$$
\frac{L^f_{i_h}}{L^f} = \frac{1}{\frac{d(s_f)}{a_T} \frac{T}{T-s_f} \sum_{j=A,M,S} \left[E_{j_m i_h} \frac{I_{j_m}}{I_{i_h}}\right] + \sum_{j=A,M,S} \left[E_{j_h i_h} \frac{I_{j_h}}{I_{i_h}}\right] + E_{l i_h} \frac{I_l}{I_{i_h}}}
\tag{86}
$$

which I can combine with Equation (80b) to condition that the wage ratio x and schooling years s_f, s_m satisfy

$$
\left(I_L \sum_{\substack{j=A_m,M_m,S_m \\ A_h,M_h,S_h,l}} \frac{E_{jS_h}}{I_{S_h}} \right)^{-1} = \left(\frac{d(s_f)}{a_T} \frac{T}{T-s_f} \sum_{j=A,M,S} \left[E_{j_m i_h} \frac{I_{j_m}}{I_{i_h}}\right] + \sum_{j=A,M,S} \left[E_{j_h i_h} \frac{I_{j_h}}{I_{i_h}}\right] + E_{l i_h} \frac{I_l}{I_{i_h}} \right)^{-1}
\tag{87}
$$

where i_h can be any non-market production sector A_h, M_h, S_h (in the Julia Code, $i_h = S_h$).

Note that for comparability of male and female hours, I set $L_m = L_f$, and require that the male time constraint holds as well:

$$
\begin{aligned}
\sum_{i=A_h,M_h,S_h,l} L^m_i + \sum_{j=A_m,M_m,S_m} L^m_j &= L^m \\
\frac{L^m_l}{L^f_{S_h}} + \sum_{i=A_h,M_h,S_h,} \frac{L^m_i}{L^f_{S_h}} + \sum_{j=A_m,M_m,S_m} \frac{L^m_j}{L^f_{S_h}} &= \frac{L^m}{L^f_{S_h}} \\
\frac{L^f_l}{L^f_{S_h}} (x\tilde{d}\tilde{H})^\eta \alpha_i^{-\eta} + \sum_{i=A_h,M_h,S_h} \frac{L^f_i}{L^f_{S_h}} \alpha_i^{-\eta} x^\eta (\tilde{d}\tilde{H})^\eta \left(\frac{T-s_f}{T-s_m}\right)^{1-\eta} & \\
+ \sum_{j=A_m,M_m,S_m} \frac{L^f_j}{L^f_{S_h}} x^\eta \alpha^{-\eta} \frac{T-s_f}{T-s_m} \frac{H(s_f)}{H(s_m)} &= \frac{L^m}{L^f_{S_h}}
\end{aligned}
\tag{88a}
$$

Defining the left-hand side as R_m, and making use of Eq. (48d) and Eq. (45f) I then get that male hours equal

$$L_i^m = \frac{L_m}{R_m} \alpha_i^{-\eta} \left(x \frac{d(s_f)H(s_f)}{d(s_m)H(s_m)} \right)^{\eta} \frac{L_i^f}{L_{S_h}^f}, \qquad i = l, \tag{88b}$$

$$L_i^m = \frac{L_m}{R_m} \alpha_i^{-\eta} \left(x \frac{d(s_f)H(s_f)}{d(s_m)H(s_m)} \right)^{\eta} \left(\frac{T - s_f}{T - s_m} \right)^{1-\eta} \frac{L_i^f}{L_{S_h}^f}, \qquad i = A_h, M_h, S_h \tag{88c}$$

and

$$L_j^m = \frac{L_m}{R_m} \alpha_i^{-\eta} x^{\eta} \left(\frac{(T - s_f)H(s_f)}{(T - s_m)H(s_m)} \right) \frac{L_j^f}{L_{S_h}^f}, \qquad j = A_m, M_m, S_m \tag{88d}$$

B.5 Deriving the conditions for schooling

Female Schooling

We can use the condition given by the first order conditions wrt female schooling and female hours in non-market production Equations (47b) and (47d) as a starting point for the derivation of a root function for the optimal level of (female) schooling years:

$$W'(s_f) + \sum_{j=A_h,M_h,S_h} a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_f} = -\lambda w_f M^f \left[d(s_g)H'(s_f) + d'(s_f)H(s_f) \right] \tag{89}$$

$$\lambda = \frac{a_T}{w_g d(s_g) H(s_g)} \frac{\delta U}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^g_{i_h}}, i = A, M, S; g = f, m; \tag{90}$$

$$W'(s_f) + \sum_{j=A_h,M_h,S_h} a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_f} = -M^f \left[\frac{H'(s_f)}{H(s_f)} + \frac{d'(s_f)}{d(s_f)} \right] \frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^f_{i_h}} a_T \tag{91}$$

Let's focus on specific parts of this equation. We have that

$$\frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^f_{i_h}} = \frac{1}{c-\bar{c}} \omega_i \left(\frac{c}{c_i}\right)^{\frac{1}{\varepsilon}} (1-\psi_i) \left(\frac{c_i}{c_{i_h}}\right)^{\frac{1}{\sigma_z}} \frac{1}{T} Z_{i_h} \xi_{i_h} (T-s_f)^{\frac{\eta-1}{\eta}} \left(L^f_{i_h}\right)^{-\frac{1}{\eta}} L_{i_h}^{\frac{1}{\eta}}$$

Since $L_{i_h} = c_{i_h} T / Z_{i_h}$,

$$\begin{aligned} \frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^f_{i_h}} &= \frac{1}{c-\bar{c}} \omega_i \left(\frac{c}{c_i}\right)^{\frac{1}{\varepsilon}} (1-\psi_i) \left(\frac{c_i}{c_{i_h}}\right)^{\frac{1}{\sigma_z}} Z_{i_h} \xi_{i_h} \left(\frac{T-s_f}{T}\right)^{\frac{\eta-1}{\eta}} \left(L^f_{i_h}\right)^{-\frac{1}{\eta}} \frac{c_{i_h}}{Z_{i_h}}^{\frac{1}{\eta}} \\ &= \frac{c}{c-\bar{c}} c^{\frac{1-\varepsilon}{\varepsilon}} c_i^{\frac{\varepsilon-\sigma_z}{\varepsilon\sigma_z}} c_{i_h}^{\frac{\sigma_z-\eta}{\sigma_z\eta}} \left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}} \omega_i (1-\psi_i) Z_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h} \left(\frac{T-s_f}{T}\right)^{\frac{\eta-1}{\eta}} \end{aligned}$$

Now let's replace c_i by c_{i_h}, c by c_i and c_{i_h} by $L^f_{i_h}$ one by one:

1. c_i by c_{i_h}: From Equation (63) we know relative demand c_{i_m}/c_{i_h}, so that we can replace c_{i_m} in the CES-aggregator for total services c_i:

$$\begin{aligned} c_{i_m} &= \left(\frac{p_{i_h}}{p_{i_m}}\right)_z^{\sigma} \left(\frac{\psi_i}{1-\psi_i}\right)^{\sigma_z} c_{i_h} \\ c_i &= \left[\psi_i \left((\frac{p_{i_h}}{p_{i_m}})_z^{\sigma} (\frac{\psi_i}{1-\psi_i})^{\sigma_z} c_{i_h} \right)^{\frac{\sigma_z-1}{\sigma_z}} + (1-\psi_i) c_{i_h}^{\frac{\sigma_z-1}{\sigma_z}} \right]^{\frac{\sigma_z}{\sigma_z-1}} \\ c_i &= c_{i_h} \left[\psi_i \left((\frac{p_{i_h}}{p_{i_m}})^{\sigma_z-1} (\frac{\psi_i}{1-\psi_i})^{\sigma_z-1} \right)^{\frac{\sigma_z-1}{\sigma_z}} + (1-\psi_i) \right]^{\frac{\sigma_z}{\sigma_z-1}} \\ c_i &= c_{i_h} \left[(1-\psi_i) \left(1 + \underbrace{(\frac{p_{i_h}}{p_{i_m}})^{\sigma_z-1} (\frac{\psi_i}{1-\psi_i})^{\sigma_z}}_{=E_{i_{mh}}} \right) \right]^{\frac{\sigma_z}{\sigma_z-1}} \end{aligned}$$

which we can simplify using Equation (64a) to

$$c_i = \left(1-\psi_i\right)^{\frac{\sigma_z}{\sigma_z-1}} \left[1+E_{i_{mh}}\right]^{\frac{\sigma_z}{\sigma_z-1}} c_{i_h} \tag{92}$$

2. c by c_i: From our computations of the marginal rates of substition, we know c_j/c_i by Equation (72):

$$\frac{c_j}{c_i} = \left(E_{j_m i_h} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{93}$$

Thus, aggregate consumption c relative to c_i equals

$$\begin{aligned}
\frac{c}{c_i} &= \left[\omega_i \sum_{j=A,M,S} \frac{\omega_j}{\omega_i} \left(\frac{c_j}{c_i}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
\frac{c}{c_i} &= \omega_i^{\frac{\varepsilon}{\varepsilon-1}} \left[\sum_{j=A,M,S} \frac{\omega_j}{\omega_i} \left(\left(E_{j_m i_m} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
\frac{c}{c_i} &= \omega_i^{\frac{\varepsilon}{\varepsilon-1}} \left[\sum_{j=A,M,S} E_{j_m i_m} \frac{E_{i_{mh}}}{E_{j_{mh}}} \frac{1+E_{j_{mh}}}{1+E_{i_{mh}}}\right]^{\frac{\varepsilon}{\varepsilon-1}}
\end{aligned} \tag{94}$$

Define

$$E_i := \frac{E_{i_{mh}}}{1+E_{i_{mh}}} \sum_{j=A,M,S} E_{j_m i_m} \frac{1+E_{j_{mh}}}{E_{j_{mh}}} \tag{95}$$

Then

$$\boxed{c = c_i \omega_i^{\frac{\varepsilon}{\varepsilon-1}} E_i^{\frac{\varepsilon}{\varepsilon-1}}} \tag{96}$$

3. Remember that $c_{i_h} = \frac{1}{T} Z_{i_h} L_{i_h}$, where Equation (50) implies that

$$c_{i_h} = Z_{i_h} \left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} L_{i_h}^f \frac{T-s_f}{T} \tag{97}$$

Now make use of the above derived Equation (96) to replace c using c_i in the second part of what is to be our condition for female schooling:

$$\frac{c}{c-\bar{c}} c^{\frac{1-\varepsilon}{\varepsilon}} c_i^{\frac{\varepsilon-\sigma_z}{\varepsilon\sigma_z}} c_{i_h}^{\frac{\sigma_z-\eta}{\sigma_z\eta}} \left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}} \omega_i(1-\psi_i) Z_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h} \left(\frac{T-s_f}{T}\right)^{\frac{\eta-1}{\eta}}$$

$$\frac{c}{c-\bar{c}} E_i^{-1} c_i^{\frac{1-\sigma_z}{\sigma_z}} c_{i_h}^{\frac{\sigma_z-\eta}{\sigma_z\eta}} \left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}} (1-\psi_i) Z_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h} \left(\frac{T-s_f}{T}\right)^{\frac{\eta-1}{\eta}} \tag{98a}$$

We then use Equation (92) to replace c_i by c_{i_h}:

$$\frac{c}{c-\bar{c}} E_i^{-1} \left((1-\psi_i)^{\frac{\sigma_z}{\sigma_z-1}} \left[1+E_{i_{mh}}\right]^{\frac{\sigma_z}{\sigma_z-1}} c_{i_h} \right)^{\frac{1-\sigma_z}{\sigma_z}} c_{i_h}^{\frac{\sigma_z-\eta}{\sigma_z\eta}} \left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}} (1-\psi_i) Z_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h} \left(\frac{T-s_f}{T}\right)^{\frac{\eta-1}{\eta}}$$

$$\frac{c}{c-\bar{c}} E_i^{-1} \left(1+E_{i_{mh}}\right)^{-1} c_{i_h}^{\frac{1-\eta}{\eta}} \left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}} Z_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h} \left(\frac{T-s_f}{T}\right)^{\frac{\eta-1}{\eta}} \tag{98b}$$

Now substitute c_{i_h} using Equation (97) yields:

$$\frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^f_{i_h}} \tag{98c}$$

$$= \frac{c}{c-\bar{c}} E_i^{-1} \left(1+E_{i_{mh}}\right)^{-1} \left(Z_{i_h} \left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{i_h} \frac{T-s_f}{T} \right)^{\frac{1-\eta}{\eta}} \left(\frac{1}{L^f_{i_h}}\right)^{\frac{1}{\eta}} Z_{i_h}^{\frac{\eta-1}{\eta}} \xi_{i_h} \left(\frac{T-s_f}{T}\right)^{\frac{\eta-1}{\eta}}$$

$$\frac{c}{c-\bar{c}} E_i^{-1} \left(1+E_{i_{mh}}\right)^{-1} \frac{I_{i_h}}{L^f_{i_h}} \tag{98d}$$

Plug into the RHS of the female schooling condition yields

$$W'(s_f) + \sum_{j=A_h,M_h,S_h} a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_f} = -a_T M^f \hat{H}(s_f) \frac{c}{c-\bar{c}} E_i^{-1} \left(1+E_{i_{mh}}\right)^{-1} \frac{I_{i_h}}{L^f_{i_h}}, \tag{99}$$

where

$$\hat{H}(s_f) := \left[\frac{H'(s_f)}{H(s_f)} + \frac{d'(s_f)}{d(s_f)}\right]. \tag{100}$$

Now check part 2 on the left hand side:

$$\sum_{j=A,M,S} a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_f} = \sum_{j=A,M,S} a_T \frac{\delta U}{\delta c} \frac{\delta c}{\delta c_j} \frac{\delta c_j}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_f}$$

$$= a_T \sum_{j=A,M,S} \frac{1}{c-\bar{c}} \omega_j \left(\frac{c}{c_j}\right)^{\frac{1}{\varepsilon}} (1-\psi_j) \left(\frac{c_j}{c_{j_h}}\right)^{\frac{1}{\sigma}} \frac{1}{T} Z_{j_h} L_j^{\frac{1}{\eta}} \xi_{j_h} (L^f_{j_h})^{\frac{\eta-1}{\eta}} (T-s_f)^{-\frac{1}{\eta}} (-1) \tag{101a}$$

Use $c_{j_h} = \frac{1}{T} Z_{j_h} L_{j_h}$ to replace L_{j_h}:

$$-a_T \sum_{j=A,M,S} \frac{1}{c-\bar{c}} \left(\frac{c}{c_j}\right)^{\frac{1}{\varepsilon}} \left(\frac{c_j}{c_{j_h}}\right)^{\frac{1}{\sigma}} \frac{1}{T} Z_{j_h} \left(\frac{T c_{j_h}}{Z_{j_h}}\right)^{\frac{1}{\eta}} \xi_{j_h} (L^f_{j_h})^{\frac{\eta-1}{\eta}} (T-s_f)^{-\frac{1}{\eta}} \omega_j (1-\psi_j)$$

$$-a_T \sum_{j=A,M,S} \frac{c}{c-\bar{c}} c^{\frac{1-\varepsilon}{\varepsilon}} c_j^{\frac{\varepsilon-\sigma}{\sigma\varepsilon}} c_{j_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} \xi_{j_h} (L^f_{j_h})^{\frac{\eta-1}{\eta}} (T-s_f)^{-\frac{1}{\eta}} \omega_j (1-\psi_j)$$

Again, use Equation (96) to replace c by c_j

$$c = c_j \omega_j^{\frac{\varepsilon}{\varepsilon-1}} E_j^{\frac{\varepsilon}{\varepsilon-1}} \tag{101b}$$

$$-a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} \left(c_j \omega_j^{\frac{\varepsilon}{\varepsilon-1}} E_j^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{1-\varepsilon}{\varepsilon}} c_j^{\frac{\varepsilon-\sigma}{\sigma\varepsilon}} c_{j_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} \xi_{j_h} (L^f_{j_h})^{\frac{\eta-1}{\eta}} (T-s_f)^{-\frac{1}{\eta}} \omega_j (1-\psi_j)$$

$$-a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1} c_j^{\frac{1-\sigma}{\sigma}} c_{j_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} \xi_{j_h} (L^f_{j_h})^{\frac{\eta-1}{\eta}} (T-s_f)^{-\frac{1}{\eta}} (1-\psi_j) \tag{101c}$$

And replace c_j by c_{j_h}:

$$c_j = (1-\psi_j)^{\frac{\sigma}{\sigma-1}} \left[1+E_{j_{mh}}\right]^{\frac{\sigma}{\sigma-1}} c_{j_h}$$

yields

$$\begin{aligned}
&-a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1} \left((1-\psi_j)^{\frac{\sigma}{\sigma-1}} \left[1+E_{j_{mh}}\right]^{\frac{\sigma}{\sigma-1}} c_{j_h}\right)^{\frac{1-\sigma}{\sigma}} \\
&\qquad \times c_{j_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} \xi_{j_h} (L^f_{j_h})^{\frac{\eta-1}{\eta}} (T-s_f)^{-\frac{1}{\eta}} (1-\psi_j) \\
&-a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1} \left(1+E_{j_{mh}}\right)^{-1} c_{j_h}^{\frac{1-\eta}{\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} \xi_{j_h} (L^f_{j_h})^{\frac{\eta-1}{\eta}} (T-s_f)^{-\frac{1}{\eta}}
\end{aligned} \tag{101d}$$

And finally replace c_{j_h} using Equation (97) :

$$c_{j_h} = Z_{j_h} \left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{j_h} \frac{T-s_f}{T}$$

$$\sum_{j=A,M,S} a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_f}$$

$$= -a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1} \left(1+E_{j_{mh}}\right)^{-1} \left(Z_{j_h} \left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}} L_{j_h}^f \frac{T-s_f}{T}\right)^{\frac{1-\eta}{\eta}}$$

$$\times \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} \xi_{j_h} (L_{j_h}^f)^{\frac{\eta-1}{\eta}} (T-s_f)^{-\frac{1}{\eta}}$$

$$= -a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1} \left(1+E_{j_{mh}}\right)^{-1} \frac{1}{T-s_f} I_{j_h}$$

$$= -\frac{a_T}{T-s_f} \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1} \left(1+E_{j_{mh}}\right)^{-1} I_{j_h} \tag{101e}$$

Putting everything together, and setting $\bar{c}$ = 0:

$$W'(s_f) - \frac{a_T}{T-s_f} \sum_{j=A,M,S} E_j^{-1} \left(1+E_{j_{mh}}\right)^{-1} I_{j_h} = -a_T \frac{M^f}{L_{i_h}^f} \hat{H}(s_f) E_i^{-1} \left(1+E_{i_{mh}}\right)^{-1} I_{i_h} \tag{102}$$

Change suffix on RHS:

$$W'(s_f) - \frac{a_T}{T-s_f} \sum_{j=A,M,S} E_j^{-1} \left(1+E_{j_{mh}}\right)^{-1} I_{j_h} = -a_T \frac{M^f}{L_{S_h}^f} \hat{H}(s_f) E_S^{-1} \left(1+E_{S_{mh}}\right)^{-1} I_{S_h}$$

which for

$$\tilde{E}_j := E_j^{-1} \left(1+E_{j_{mh}}\right)^{-1} I_{j_h}. \tag{103}$$

simplifies to

$$W'(s_f) - \frac{a_T}{T-s_f} (\tilde{E}_A + \tilde{E}_M + \tilde{E}_S) = -a_T \frac{M^f}{L_{S_h}^f} \hat{H}(s_f) \tilde{E}_S. \tag{104}$$

or

$$W'(s_f) = a_T \left(\frac{1}{T-s_f} (\tilde{E}_A + \tilde{E}_M + \tilde{E}_S) - \frac{M^f}{L_{S_h}^f} \hat{H}(s_f) \tilde{E}_S \right) \tag{105}$$

Male Schooling

On the male side,

$$W'(s_m) + \sum_{j=A_h,M_h,S_h} a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_m} = -M^m \left[\frac{H'(s_m)}{H(s_m)} + \frac{d'(s_m)}{d(s_m)}\right] \frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^m_{i_h}} a_T$$

Note the derivative $\frac{\delta c_{i_h}}{\delta L^m_{i_h}}$, which takes the form

$$\frac{\delta c_{i_h}}{\delta L^m_{i_h}} = \frac{1}{T} Z_{i_h}(1-\xi_{i_h})(T-s_m)^{\frac{\eta-1}{\eta}} \left(\frac{L_{i_h}}{L^m_{i_h}}\right)^{\frac{1}{\eta}} \tag{106a}$$

$$= \frac{1}{T} Z_{i_h}(1-\xi_{i_h})(T-s_m)^{\frac{\eta-1}{\eta}} \left(\frac{L_{i_h}}{\alpha_{i_h}^{-\eta} x^\eta (\tilde{d}\tilde{H})^\eta \tilde{T}^{1-\eta} L^f_{i_h}}\right)^{\frac{1}{\eta}} \text{ by Equation (48d)} \tag{106b}$$

$$= \frac{1}{T} Z_{i_h}(1-\xi_{i_h}) \left(\frac{L_{i_h}}{L^f_{i_h}}\right)^{\frac{1}{\eta}} \alpha_{i_h} \left(x\tilde{d}\tilde{H}\right)^{-1} \left(T-s_f\right)^{\frac{\eta-1}{\eta}}$$

$$= \frac{1}{T} Z_{i_h}(1-\xi_{i_h}) \left(\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}} (T-s_f)\right)^{\frac{1}{\eta}} \alpha_{i_h} \left(x\tilde{d}\tilde{H}\right)^{-1} \left(T-s_f\right)^{\frac{\eta-1}{\eta}} \text{ by Equation (50)}$$

$$= \frac{1}{T} Z_{i_h} \left(\left(\frac{\xi_{i_h}}{I_{i_h}}\right)^{\frac{\eta}{\eta-1}}\right)^{\frac{1}{\eta}} \xi_{i_h} \left(x\tilde{d}\tilde{H}\right)^{-1} \left(T-s_f\right)$$

$$= \underbrace{Z_{i_h} \left(\frac{1}{I_{i_h}}\right)^{\frac{1}{\eta-1}} \left(\xi_{i_h}\right)^{\frac{\eta}{\eta-1}} \frac{T-s_f}{T}}_{\text{Known from eq. (53a)}} \left(x\tilde{d}\tilde{H}\right)^{-1}$$

$$= \frac{\delta c_{i_h}}{\delta L^f_{i_h}} \left(x\tilde{d}\tilde{H}\right)^{-1} \tag{106c}$$

such that we get

$$-a_T M^m \hat{H}(s_m) \frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^m_{i_h}} = -a_T M^m \hat{H}(s_m) \frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^m_{i_h}}$$

$$= -a_T M^m \hat{H}(s_m) \frac{1}{c-\bar{c}} \frac{\delta c}{\delta c_i} \frac{\delta c_i}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta L^f_{i_h}} \left(x\tilde{d}\tilde{H}\right)^{-1}$$

$$= -a_T M^m \hat{H}(s_m) \frac{c}{c-\bar{c}} E_i^{-1} \left(1+E_{i_{mh}}\right)^{-1} \frac{I_{i_h}}{L^f_{i_h}} \left(x\tilde{d}\tilde{H}\right)^{-1}$$

$$= -a_T \frac{M^m}{L^f_{S_h}} \hat{H}(s_m) \frac{c}{c-\bar{c}} \tilde{E}_S \left(x\tilde{d}\tilde{H}\right)^{-1} \tag{106d}$$

where the final step uses the definition for $\tilde{E}_i$ and $i = S$.

The second part on the LHS equals:

$$\sum_{j=A_h,M_h,S_h} a_T \frac{\delta U}{\delta c_{j_h}} \frac{\delta c_{j_h}}{\delta s_m}$$
$$= -a_T \sum_{j=A,M,S} \frac{1}{c-\bar{c}} \omega_j \left(\frac{c}{c_j}\right)^{\frac{1}{\varepsilon}} (1-\psi_j) \left(\frac{c_j}{c_{j_h}}\right)^{\frac{1}{\sigma}} \frac{1}{T} Z_{j_h} L_j^{\frac{1}{\eta}} (1-\xi_{j_h})(L_{j_h}^m)^{\frac{\eta-1}{\eta}} (T-s_m)^{-\frac{1}{\eta}} \tag{107}$$

The following three steps are similar to the derivations for women. Since $c_{j_h} = \frac{1}{T} Z_{j_h} L_{j_h}$, replace L_j:

$$= -a_T \sum_{j=A,M,S} \frac{1}{c-\bar{c}} \left(\frac{c}{c_j}\right)^{\frac{1}{\varepsilon}} \left(\frac{c_j}{c_{j_h}}\right)^{\frac{1}{\sigma}} \frac{1}{T} Z_{j_h} \left(\frac{c_{j_h} T}{Z_{j_h}}\right)^{\frac{1}{\eta}} (1-\xi_{j_h})(L_{j_h}^m)^{\frac{\eta-1}{\eta}} (T-s_m)^{-\frac{1}{\eta}} \omega_j (1-\psi_j)$$
$$= -a_T \sum_{j=A,M,S} \frac{c}{c-\bar{c}} c^{\frac{1-\varepsilon}{\varepsilon}} c_j^{\frac{\varepsilon-\sigma}{\sigma\varepsilon}} c_{j_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} (1-\xi_{j_h})(L_{j_h}^m)^{\frac{\eta-1}{\eta}} (T-s_m)^{-\frac{1}{\eta}} \omega_j (1-\psi_j)$$

Again, use Equation (96) to replace c by c_j from $c = c_j \omega_j^{\frac{\varepsilon}{\varepsilon-1}} E_j^{\frac{\varepsilon}{\varepsilon-1}}$ gives

$$-a_T \sum_{j=A,M,S} \frac{c}{c-\bar{c}} \left[c_j \omega_j^{\frac{\varepsilon}{\varepsilon-1}} E_j^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{1-\varepsilon}{\varepsilon}} c_j^{\frac{\varepsilon-\sigma}{\sigma\varepsilon}} c_{j_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}}$$
$$\times (1-\xi_{j_h})(L_{j_h}^m)^{\frac{\eta-1}{\eta}} (T-s_m)^{-\frac{1}{\eta}} \omega_j (1-\psi_j)$$
$$= -a_T \sum_{j=A,M,S} \frac{c}{c-\bar{c}} E_j^{-1} c_j^{\frac{1-\sigma}{\sigma}} c_{j_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} (1-\xi_{j_h})(L_{j_h}^m)^{\frac{\eta-1}{\eta}} (T-s_m)^{-\frac{1}{\eta}} (1-\psi_j) \tag{108}$$

Replace c_j by c_{j_h} from $c_j = \left(1-\psi_j\right)^{\frac{\sigma}{\sigma-1}} \left[1+E_{j_{mh}}\right]^{\frac{\sigma}{\sigma-1}} c_{j_h}$ gives

$$-a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1} \left[\left(1-\psi_j\right)^{\frac{\sigma}{\sigma-1}} \left[1+E_{j_{mh}}\right]^{\frac{\sigma}{\sigma-1}} c_{j_h}\right]^{\frac{1-\sigma}{\sigma}} c_{j_h}^{\frac{\sigma-\eta}{\sigma\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}}$$
$$\times (1-\xi_{j_h})(L_{j_h}^m)^{\frac{\eta-1}{\eta}} (T-s_m)^{-\frac{1}{\eta}} (1-\psi_j)$$
$$= -a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1} (1+E_{j_{mh}})^{-1} c_{j_h}^{\frac{1-\eta}{\eta}} \left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}} \left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} (1-\xi_{j_h})(L_{j_h}^m)^{\frac{\eta-1}{\eta}} (T-s_m)^{-\frac{1}{\eta}} \tag{109}$$

And finally replace c_{j_h} using Equation (97) for $c_{j_h} = Z_{j_h}\left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{j_h}\frac{T-s_f}{T}$

$$
\begin{aligned}
& -a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1}(1+E_{j_{mh}})^{-1}\left[Z_{j_h}\left(\frac{\xi_{j_h}}{I_{j_h}}\right)^{\frac{\eta}{\eta-1}} L^f_{j_h}\frac{T-s_f}{T}\right]^{\frac{1-\eta}{\eta}}\left(\frac{1}{T}\right)^{\frac{\eta-1}{\eta}}\left(Z_{j_h}\right)^{\frac{\eta-1}{\eta}} \\
& \quad \times(1-\xi_{j_h})(L^m_{j_h})^{\frac{\eta-1}{\eta}}(T-s_m)^{-\frac{1}{\eta}} \\
& = -a_T \frac{c}{c-\bar{c}} \sum_{j=A,M,S} E_j^{-1}(1+E_{j_{mh}})^{-1} I_{j_h}(T-s_f)^{\frac{1-\eta}{\eta}} \alpha_{j_h}^{-1}\left(\frac{L^m_{j_h}}{L^f_{j_h}}\right)^{\frac{\eta-1}{\eta}}(T-s_m)^{-\frac{1}{\eta}} \\
& = -a_T \frac{c}{c-\bar{c}}(T-s_f)^{\frac{1-\eta}{\eta}}(T-s_m)^{-\frac{1}{\eta}} \sum_{j=A,M,S} \tilde{E}_j \alpha_{j_h}^{-1}\left(\frac{L^m_{j_h}}{L^f_{j_h}}\right)^{\frac{\eta-1}{\eta}} \qquad (110)
\end{aligned}
$$

Replace male by female hours from Equation (48d): $\frac{L^m_{j_h}}{L^f_{j_h}} = \alpha_{j_h}^{-\eta} x^{\eta} (\tilde{dH})^{\eta} \tilde{T}^{1-\eta}$

$$
\begin{aligned}
& -a_T \frac{c}{c-\bar{c}}(T-s_f)^{\frac{1-\eta}{\eta}}(T-s_m)^{-\frac{1}{\eta}} \sum_{j=A,M,S} \tilde{E}_j \alpha_{j_h}^{-1}\left(\alpha_{j_h}^{-\eta} x^{\eta}(\tilde{dH})^{\eta}\tilde{T}^{1-\eta}\right)^{\frac{\eta-1}{\eta}} \\
& = -a_T \frac{c}{c-\bar{c}} \frac{1}{T-s_m} \sum_{j=A,M,S} \tilde{E}_j \alpha_{j_h}^{-\eta}\left(x\tilde{dH}\right)^{\eta-1}\tilde{T}^{1-\eta} \qquad (111)
\end{aligned}
$$

Change suffix on LHS, set $\bar{c} = 0$:

$$
\sum_{i=A_h,M_h,S_h} a_T \frac{\delta U}{\delta c_{i_h}} \frac{\delta c_{i_h}}{\delta s_m} = -\frac{a_T}{T-s_m} \sum_{i=A,M,S} \tilde{E}_i \alpha_{i_h}^{-\eta}\left(x\tilde{dH}\right)^{\eta-1}\tilde{T}^{1-\eta} \qquad (112)
$$

The condition then becomes

$$
\boxed{W'(s_m) - \frac{a_T}{T-s_m} \sum_{i=A,M,S} \tilde{E}_i \alpha_{i_h}^{-\eta}\left(x\tilde{dH}\right)^{\eta-1}\tilde{T}^{1-\eta} = -a_T \frac{M^m}{L^f_{S_h}} \hat{H}(s_m)\tilde{E}_S\left(x\tilde{dH}\right)^{-1}} \qquad (113)
$$

Or noting that $\alpha_{i_h}^{-\eta}\left(x\tilde{dH}\right)^{\eta-1}\tilde{T}^{1-\eta} = \frac{1-I_{i_h}}{I_{i_h}}$:

$$
\boxed{W'(s_m) - \frac{a_T}{T-s_m} \sum_{i=A,M,S} \tilde{E}_i \frac{1-I_{i_h}}{I_{i_h}} = -a_T \frac{M^m}{L^f_{S_h}} \hat{H}(s_m)\tilde{E}_S\left(x\tilde{dH}\right)^{-1}} \qquad (114)
$$

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